

## Problem Set 2 Discussion and Solutions

We can begin with a simple integral analysis for the problem, noting that the second of the two equilibrium equations in the "axial" equation and the first the "vertical" one. Thus we want the integral of each equation and the moment of the second. Let

$$F = \int_{R-\frac{h}{2}}^{R+\frac{h}{2}} \sigma_{\theta} dr, \quad V = \int_{R-\frac{h}{2}}^{R+\frac{h}{2}} \tau_{r\theta} dr, \quad M = \int_{R-\frac{h}{2}}^{R+\frac{h}{2}} (R-r)\sigma_{\theta} dr$$

be the obvious analogs of the corresponding terms in the straight beam problem. The two force equations and one moment equation can be seen to be<sup>1</sup>

$$\frac{dF}{d\theta} + V = 0, \quad \frac{dV}{d\theta} - F = -R_2\sigma_{r2} + R_1\sigma_{r1} \approx -Rq, \quad \frac{dM}{d\theta} + RV = 0$$

where  $R$  denotes the radius of the middle surface (line),  $R_2$  and  $R_1$  denote the outer and inner radii, respectively, and  $q$  denotes the pressure on the inner face of the beam for the case that this is the only stress, and the approximation is the difference between  $R$  and  $R_1$ , here small. These are quite different from the straight beam equations. The major reason is the appearance of the axial normal stress in the radial equilibrium equation. We can eliminate  $F$  and write two equations for  $V$  and  $M$ , viz.

$$\frac{d^2V}{d\theta^2} + V = -R\frac{dq}{d\theta}, \quad \frac{dM}{d\theta} + RV = 0$$

and the solutions to these are quite different depending on whether  $q$  is constant or not. In either case  $V$  has a homogeneous solution

$$V_H = A \sin\theta + B \cos\theta$$

If  $q$  is constant the particular solution for  $V$  is zero. The moment can be obtained by integration

$$M = M_0 - AR \cos\theta + BR \sin\theta$$

---

<sup>1</sup>Multiply each equation by  $r$  and integrate to get the first two. Multiply the second equation by  $r$  times  $R - r$  and integrate to get the third.

and the axial force is equal to

$$F = qR + A \cos\theta - B \sin\theta$$

Define  $\theta = 0$  to be straight up and let the beam extend from  $-\theta_0$  to  $+\theta_0$ . The zero moment condition at  $\theta = \pm\theta_0$  gives  $M_0 = AR \cos\theta_0$  and  $B = 0$ . The horizontal and vertical force balances give  $A = -qR \cos\theta_0$ . To summarize, the resultants are:

$$F = qR (1 - \cos\theta_0 \cos\theta), \quad V = -qR \cos\theta_0 \sin\theta, \quad M = -qR^2 \cos\theta_0 (\cos\theta_0 - \cos\theta)$$

The most obvious difference between this solution and the straight beam is that  $F$  is not zero. The moment is also nonzero, so  $\sigma_\theta$  must have both odd and even components in the local radial direction. If we follow the rule of simplest stress components that make the integrals come out right, we'll get

$$\sigma_\theta = \frac{q}{\varepsilon} (1 - \cos\theta_0 \cos\theta) - 12 \frac{q}{\varepsilon^2} \cos\theta_0 (\cos\theta_0 - \cos\theta) \zeta$$

$$\tau_{r,\theta} = -6 \frac{q}{\varepsilon} \cos\theta_0 \sin\theta \left( \frac{1}{4} - \zeta^2 \right)$$

where  $\zeta$  is defined as in the problem set. Substitution of this into the radial equilibrium equation leads to an expression for the radial stress

$$\sigma_r = -6 \frac{q}{\varepsilon} \cos\theta_0 (\cos\theta_0 - \cos\theta) \left( \frac{1}{4} - \zeta^2 \right) - q \left( \frac{1}{2} - \zeta \right) \left( 2 \cos\theta_0 (3 \cos\theta_0 - 4 \cos\theta) \zeta \left( \zeta + \frac{1}{2} \right) + 1 \right)$$

where the first term comes from the moment part of the axial stress, and vanishes on both surfaces. The second term is the usual normal stress that matches the pressure applied to the inner surface. This set of stresses satisfies the equilibrium equations to lowest order (the  $O(1/\varepsilon)$  and the  $O(1)$  term of the radial equilibrium equation and the  $O(1/\varepsilon)$  term of the azimuthal equilibrium equation). I will leave it to you to deduce the displacements that go along with these stresses. Note that they are approximations, and that the idea of thinness enters here to allow us to choose forms for the radial ( $\zeta$ ) dependence of the stress components.

How does this connect to the equations we would deduce using the scaling arguments — the question posed in the problem set? The equilibrium equations simplify under the suggested coordinate transformation, giving

$$\frac{\partial \sigma_r}{\partial \zeta} - \varepsilon \sigma_\theta = 0, \quad \frac{\partial \tau_{r,\theta}}{\partial \zeta} + \varepsilon \frac{\partial \sigma_\theta}{\partial \theta} = 0$$

We see that the shear stress is small compared to the azimuthal normal stress, but that the radial normal stress is only  $\varepsilon$  smaller than the azimuthal stress, rather than  $\varepsilon^2$  smaller as in the case of the straight beam. If we scale the radial stress by the applied pressure,  $q$  and examine the strain equations, we find  $u$  and  $v$  scales to be

$$U = \frac{q}{E}, \quad V = \frac{q}{\varepsilon E}$$

and the three strain equations reduce to

$$\frac{\partial u}{\partial \zeta} = -\frac{v}{E} \sigma_\theta, \quad \frac{\partial v}{\partial \theta} = \frac{R}{E} \sigma_\theta, \quad \frac{\partial v}{\partial \zeta} = 0$$

This means that  $v$  is independent of  $\zeta$  and so too is  $\sigma_\theta$ . The second equilibrium equation gives

$$\tau_{r\theta} = -\frac{q}{E} \frac{\partial^2 v}{\partial \theta^2} \zeta + \text{constant}$$

and this cannot satisfy its boundary conditions unless  $v$  is linear in  $\theta$  and  $\sigma_\theta$  is independent of  $\theta$ . In that case  $\tau_{r\theta}$  is identically zero. There is no  $V$  and no  $M$ . The solution can be summarized as

$$u = -v \frac{qR}{E} \zeta + u_0, \quad v = \frac{qR}{\varepsilon E} \theta + v_0, \quad \sigma_\theta = \frac{q}{\varepsilon}, \quad \sigma_r = q \left( \zeta - \frac{1}{2} \right), \quad \tau_{r\theta} = 0$$

The pressure inside is balanced by a hoop stress analog. There are no moments and no shear flow. We cannot satisfy displacement boundary conditions at the ends. The axial displacement increases from some reference point. The radial displacement is negative outside the middle line and positive inside, thinning where the radius increases and fattening where it decreases. This is a geometrically plausible result. This is essentially equivalent to the *membrane approximation* of shell theory, and we learn that we cannot always get what we want by a simple expansion.