Beam Bending by Scale Analysis: An illustration of scaling

The idea of scaling is to combine physical intuition with mathematics to identify what terms are important physically, and to identify the probable size of the errors that stem from the approximations made. We know that beam theory requires that the beam be much longer than it is high or wide, and that the deformations must be small enough that linear elasticity is appropriate — the small strain approximation. These are different approximations; we will assess both in this note. The point of this note is to outline the derivation of beam theory from three dimensional elasticity. I take the problem to be one of plane stress *a priori*, so that

$$\tau_{zx} = \tau_{zy} = \sigma_z = 0, \quad \frac{\partial}{\partial z}(\cdots) = 0$$

Scaling means to choose representative values for the variables of the problem, both dependent and independent. These scaling parameters are constants taken from the formulation of the problem. Sometimes they are obvious; sometimes they need to be found from the analysis of the problem. For this problem, we can select the beam length, L as appropriate for the x coordinate, its height, h for the y coordinate, and a measure of the load for the vertical normal stress σ_y. We do this by the following transformations

$$x = L\,\xi, \quad y = h\eta, \quad \sigma_y = \frac{P}{wL}\mathrm{s}_y$$

where s_y denotes the scaled (dimensionless) stress, P the total load on the beam and w its width. The integrals over the height of the beam become

$$\int_{-h/2}^{h/2} \cdots\, dy \rightarrow h\int_{-1/2}^{1/2} \cdots\, d\eta, \quad \int_{-h/2}^{h/2} y \cdots\, dy \rightarrow h^2\int_{-1/2}^{1/2} \eta \cdots\, d\eta$$

The scales for the other stress components can be deduced from the equilibrium equations, which may be written in terms of the scaled independent variables ξ and η as

$$\frac{1}{L}\frac{\partial\sigma_x}{\partial\xi} + \frac{1}{h}\frac{\partial\tau_{xy}}{\partial\eta} = 0, \quad \frac{1}{L}\frac{\partial\tau_{yx}}{\partial\xi} + \frac{1}{h}\frac{\partial\sigma_y}{\partial\eta} = 0$$

from which it follows (how? convince yourself of this) that an appropriate set of scales will be

$$\tau_{xy} = \frac{P}{\varepsilon wL}\mathrm{t}_{xy}, \quad \sigma_x = \frac{P}{\varepsilon^2 wL}\mathrm{s}_x$$

where the new stress components are dimensionless and $\varepsilon = h/L \ll 1$. The typical shear stress is $1/\varepsilon$ times the typical transverse normal stress, and the axial normal stress is another factor of $1/\varepsilon$ larger still.

That ε is small suggests that the problem may have a simple form that takes advantage of this smallness. Clearly the solution to the problem depends on ε. It may happen that this dependence can be made explicit by supposing the dependent variables to have an expansion in powers of ε, and hoping that the leading terms in these power series will be a good representation of the actual solution, and that errors can be estimated by the size of the terms that have not yet been found. (The process is more complicated than this. This general area of study is called *perturbation theory*, and there many books written about it. I want to use these ideas to connect three dimensional elasticity to our plate and shell theory without worrying too much about the theoretical details.) Thus I assume that

$$\mathrm{s}_x = \mathrm{s}_x^{(0)} + \varepsilon \mathrm{s}_x^{(1)} + \varepsilon^2 \mathrm{s}_x^{(2)} + \cdots, \quad \mathrm{s}_y = \mathrm{s}_y^{(0)} + \varepsilon \mathrm{s}_y^{(1)} + \varepsilon^2 \mathrm{s}_y^{(2)} + \cdots, \quad \mathrm{t}_{xy} = \mathrm{t}_{xy}^{(0)} + \varepsilon \mathrm{t}_{xy}^{(1)} + \varepsilon^2 \mathrm{t}_{xy}^{(2)} + \cdots$$

The lowest order realization of the equilibrium equations is unchanged in form, and may be written

$$\frac{\partial \mathrm{s}_x^{(0)}}{\partial \xi} + \frac{\partial \mathrm{t}_{xy}^{(0)}}{\partial \eta} = 0, \quad \frac{\partial \mathrm{t}_{yx}^{(0)}}{\partial \xi} + \frac{\partial \mathrm{s}_y^{(0)}}{\partial \eta} = 0$$

The integrated equilibrium equations are unchanged, and we get three equations — two forces

$$\frac{d}{d\xi}\int_{-1/2}^{1/2} \mathrm{s}_x^{(0)} d\eta = \frac{dF^{(0)}}{d\xi} = 0, \quad \frac{d}{d\xi}\int_{-1/2}^{1/2} \mathrm{t}_{xy}^{(0)} d\eta + \mathrm{q}^{(0)} = \frac{d\mathrm{V}^{(0)}}{d\xi} + \mathrm{q}^{(0)} = 0$$

and one moment

$$\frac{d}{d\xi}\int_{-1/2}^{1/2} \eta \mathrm{s}_x^{(0)} d\eta + \int_{-1/2}^{1/2} \frac{\partial}{\partial \eta}\left(\eta \mathrm{t}_{xy}^{(0)}\right) d\eta - \int_{-1/2}^{1/2} \mathrm{t}_{xy}^{(0)} d\eta = -\frac{d\mathrm{M}^{(0)}}{d\xi} - \mathrm{V}^{(0)} = 0$$

where these are the usual equations made dimensionless: $\mathrm{q}^{(0)}$, $\mathrm{V}^{(0)}$ and $\mathrm{M}^{(0)}$ denote the leading terms in the dimensionless load[1], shear and moment, respectively.

The interesting part begins when we take a look at the strains and displacements. The strains may be written in terms of the stresses as

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[1] The load is not typically expanded, so that this term is typically the entire load function without approximation.

$$e_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}, \quad e_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E}, \quad \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{3\lambda + 2\mu}{\lambda + \mu}\frac{\tau_{xy}}{E} = c_1\frac{\tau_{xy}}{E}$$

where E denotes Young's modulus, ν Poisson's ratio and λ and μ the Lamé constants. (This form is a little more convenient for the scaling analysis than an expression in terms of the Lamé constants only.) Note that $0 < \nu < 1/2$ and $2 < c_1 < 3$; only E affects scaling arguments. If we scale these equations denoting the dimensionless x and y displacements by u and v, respectively, and their scales by U and V ($u_x = Uu$ and $u_y = Vv$), we obtain

$$e_x = \frac{U}{L}\frac{\partial u}{\partial \xi} \approx \frac{P}{\varepsilon^2 wLE}s_x^{(0)}, \quad e_y = \frac{V}{h}\frac{\partial v}{\partial \eta} \approx -\nu\frac{P}{\varepsilon^2 wLE}s_x^{(0)}, \quad \gamma_{xy} = \frac{U}{h}\frac{\partial u}{\partial \eta} + \frac{V}{L}\frac{\partial v}{\partial \xi} = \frac{3\lambda + 2\mu}{\lambda + \mu}\frac{P}{\varepsilon wLE}t_{xy}^{(0)}$$

The strains have to be small, so we learn immediately that there is a load limit for this problem

$$P \ll \varepsilon^2 wLE$$

The first equation determines the appropriate scale for axial displacement

$$U = \frac{P}{\varepsilon^2 wE} \ll L$$

The question of transverse displacement is a little more complicated. The two terms in the second equation can be balanced if $V = \varepsilon U$. This, however, leaves the third equation incapable of being balanced unless the axial displacement is independent of η at lowest order, which is not possible if the beam is to bend. Therefore, we choose $V = U/\varepsilon$ to balance the third equation, and replace the second equation by the condition that v be independent of η to lowest order, which is exactly what we expect from beam theory. To summarize

$$u_x = \frac{P}{\varepsilon^2 wE}\left(u^{(0)} + \varepsilon u^{(1)} + \dots\right), \quad u_y = \frac{P}{\varepsilon^3 wE}\left(v^{(0)} + \varepsilon v^{(1)} + \dots\right)$$

and the leading terms satisfy the following three differential equations

$$\frac{\partial u^{(0)}}{\partial \xi} = s_x^{(0)}, \quad \frac{\partial v^{(0)}}{\partial \eta} = 0, \quad \frac{\partial u^{(0)}}{\partial \eta} + \frac{\partial v^{(0)}}{\partial \xi} = 0$$

The axial displacement may be eliminated from the first and third of these

$$\frac{\partial^2 v^{(0)}}{\partial \xi^2} + \frac{\partial s_x^{(0)}}{\partial \eta} = 0$$

This is the moment curvature relation, although it will take a little algebra to demonstrate that fact. The first term in the equation is independent of η, so the second must be as well. The dimensionless axial stress is then proportional to η, and in that case must be equal to the stress <u>assumed</u> in beam theory. Thus

$$\sigma_x \approx -\frac{M_b}{I}y \;\rightarrow \mathrm{s}_x^{(0)} = -\frac{M_b}{I}h\eta\,\frac{wL\varepsilon^2}{P}$$

We may also write

$$v^{(0)} \approx \frac{\varepsilon^3 wL}{P}v$$

so that the candidate moment curvature relation becomes

$$IE\frac{\partial^2 v}{\partial x^2} = M_b$$

as required.

The terms neglected in each approximation have been ε^2 smaller than those retained, and it is reasonable to guess that the results obtained have relative errors of the order of ε^2. To establish this would require us to find the correction terms, which is well beyond the scope of this exercise.