

## Summary of the Plate Equations

We derived a set of plate equations based on  $\varepsilon = h/L \ll 1$ . This reduced to a single equation governing the transverse displacement,  $w(x,y)$

$$D \Delta (\Delta w) = p(x,y) \quad (1)$$

where  $\Delta$  denotes the (two-dimensional) scalar Laplacian. This equation is in dimensional form, and the physical parameter

$$D = \frac{Eh^3}{12(1 - \nu^2)}$$

is called the flexural rigidity. The plate problem is thus reduced to solving the inhomogeneous biharmonic equation subject to an appropriate set of boundary conditions. In order to apply and assess this, we need the other dependent variables in terms of  $w$ . From this we can assess the result and work out the appropriate boundary conditions. It is important to remember that the governing equation and all the dependent variables and boundary conditions we have deduced depend on the smallness of  $\varepsilon$  and are not applicable to all solid objects, even if they are flat.

The equilibrium equations are unchanged in form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (2a-2c)$$

but the strain equations (to lowest order) are different (and simpler), viz.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{E} \sigma_x - \frac{\nu}{E} \sigma_y, & \frac{\partial v}{\partial y} &= -\frac{\nu}{E} \sigma_x + \frac{1}{E} \sigma_y, & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \frac{1 + \nu}{E} \tau_{xy} \\ \frac{\partial w}{\partial z} &= 0, & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} &= 0, & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} &= 0 \end{aligned} \quad (3a-3f)$$

where  $u$  and  $v$  denote the  $x$  and  $y$  components of the displacement, respectively. The key change is that  $w$  is independent of  $z$  (from (3d)), so that  $u$ ,  $v$ , and the in-plane normal stresses are all linear in  $z$ .

Let the origin of  $z$  be the neutral surface, typically midway between the two face surfaces.

Then the transverse displacements may be obtained from (3e) and (3f):

$$u = -\frac{\partial w}{\partial x}z, \quad v = -\frac{\partial w}{\partial y}z \quad (4a-4b)$$

Equations (3a)-(3c) may then be solved for the three in-plane stresses:

$$\begin{aligned} \sigma_x &= -\frac{E}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)z, & \sigma_y &= -\frac{E}{(1-\nu^2)}\left(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)z \\ \tau_{xy} &= -\frac{E}{(1+\nu)}\frac{\partial^2 w}{\partial x \partial y}z \end{aligned} \quad (5a-5c)$$

Equations (2a)-(2c) may then be integrated to obtain the remaining three stress components:

$$\begin{aligned} \tau_{xz} &= -\frac{E}{2(1-\nu^2)}\frac{\partial \Delta w}{\partial x}\left(\left(\frac{h}{2}\right)^2 - z^2\right), & \tau_{yz} &= -\frac{E}{2(1-\nu^2)}\frac{\partial \Delta w}{\partial y}\left(\left(\frac{h}{2}\right)^2 - z^2\right) \\ \sigma_z &= \frac{E}{6(1-\nu^2)}\Delta(\Delta w)\left(2\left(\frac{h}{2}\right)^3 + 3\left(\frac{h}{2}\right)^2 z - z^3\right) \end{aligned} \quad (6a-6c)$$

Note that when  $z = h/2$ ,  $\sigma_z = p(x,y)$  as required. The force and moment resultants may be found by direct integration. The in-plane forces are zero:

$$N_x = 0, \quad N_y = 0, \quad N_{xy} = 0 \quad (7a-7c)$$

The transverse forces are given by

$$Q_x = -D\frac{\partial \Delta w}{\partial x}, \quad Q_y = -D\frac{\partial \Delta w}{\partial y} \quad (8a-8b)$$

which are the same as Gould's equations (10.9a,b), and the moments are given by

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right), \quad M_y = -D\left(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \quad M_{xy} = -(1+\nu)D\frac{\partial^2 w}{\partial x \partial y} \quad (9a-9c)$$

which are the same as Gould's equations (10.8a-c).

There is a hidden assumption in all of this, which is that there are no in-plane stresses applied at the boundaries. It is straightforward to verify that the following set of stresses and displacements satisfy the reduced equations (2) and (3):

$$\sigma_x = \sigma_{10}, \quad \sigma_y = \sigma_{20}, \quad u = \frac{1}{E}(\sigma_{10} - \nu\sigma_{20})(x - x_0), \quad v = \frac{1}{E}(-\nu\sigma_{10} + \sigma_{20})(y - y_0) \quad (10a-10d)$$

where the stresses are constant and the reference positions  $x_0$  and  $y_0$  are typically taken to be either at the edges if motion is permitted at the edges, or in the center. Equations (2) and (3) are linear, so these stresses and displacements can be added to any solution by the principle of superposition. They are required to prevent the vertical ( $w$ ) deformation of a plate from pulling the edges closer together if the neutral surface does not stretch. The strains associated with these linear displacements provide the required stretching. This issue is apparently not discussed in Gould, but it does appear in Timoshenko, and we will discuss this in class. The  $x$  strain is given by

$$e_x = \frac{L' - L}{L} = \frac{1}{L} \int_0^L \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2} dx - 1 \approx \frac{1}{2L} \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \quad (11a)$$

where  $L$  denotes the extent of the plate in the  $x$  direction and  $L'$  the length of the neutral axis line after deformation, and the last approximation comes from the fact that the gradient of  $w$  must be small. The associated stress  $\sigma_{10}$  is proportional to

$$\frac{E}{2L} \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \quad (11b)$$

where the proportionality constant depends on the  $y$  state of stress. For one dimensional bending it is simply  $1/(1 - \nu^2)$ . The expressions for the  $y$  components are given by the obvious analogs.

The force and moment resultants can be used to deduce the homogeneous boundary conditions for equation (1) based on the standard boundary condition for edge loading of a plate: clamped, simply supported or free. The results are summarized by Gould in his table 10-1 (p. 296). I copy that here for convenient reference.

Condition	$x = a$	$y = b$
fixed/clamped	$w = 0 = \frac{\partial w}{\partial x}$	$w = 0 = \frac{\partial w}{\partial y}$
simply supported	$w = 0 = \frac{\partial^2 w}{\partial x^2}$	$w = 0 = \frac{\partial^2 w}{\partial y^2}$
free	$\frac{\partial \Delta w}{\partial x} = 0, \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) = 0$	$\frac{\partial \Delta w}{\partial y} = 0, \left( -\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0$

Here  $a$  and  $b$  denote specific values of  $x$  and  $y$  respectively, on which the boundary conditions are to be imposed. It is worth noting that plates can be loaded at their boundaries as well as, or instead of, being loaded on their surfaces. In that case the boundary conditions will be inhomogeneous. We will discuss this in class. The in-plane normal stress can be found *a posteriori*.