SUMMARY OF LINEAR ELASTICITY

In linear elasticity the components of the strain tensor are related linearly to derivatives of the displacement components, and the components of the stress tensor related linearly to the components of the strain tensor. The elements of the stress tensor represent forces per unit area. The equations of static equilibrium are obtained from the divergence of the stress tensor. This page deals with some simple aspects of this and provides some useful formulae.

The stress tensor will be written as σ_{ij}, and the symbol will also stand for the individual components. The first index indicates a surface and the second a direction, so that σ_{ij} denotes the force per unit area on the ith face in the jth direction. (Specifically, σ_{xy} denotes the force per unit area on a surface normal to the x axis pointing in the y direction.) The relation between stress and strain is written

$$\sigma_{ij} = 2\mu e_{ij} + \lambda\Theta\delta_{ij}$$

where λ and μ denote the Lamé constants of elasticity (related to the engineering moduli below), e_{ij} the strain tensor, Θ the dilation, and

$$\delta_{ij} = \begin{matrix} 1, & i = j \\ 0, & i \neq j \end{matrix}$$

denotes the Kronecker delta. The dilation may be written in terms of the displacement vector $\mathbf{u}$ as $\mathbf{div}(\mathbf{u})$. It may also be written in terms of the principle components of the strain tensor:

$$\Theta = e_{11} + e_{22} + e_{33}$$

The equations of equilibrium may be written entirely in terms of the displacement vector. The resulting equations are called the Navier equations of elasticity. In their steady state form they are

$$0 = (\lambda + 2\mu)\mathbf{grad}(\mathbf{div\ u}) - \mu\ \mathbf{curl}(\mathbf{curl\ u})$$

It would be nice to write out all the various relations among displacement, strain and stress in general, but this is impossible without a serious digression into tensor analysis, which is beyond the scope of this note. The Cartesian coordinate relations are simple, and I will write them out, and then write out the corresponding equations in cylindrical and spherical coordinates as useful special cases. The components of strain are given in terms of the components of displacement by

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y}, \quad e_{zz} = \frac{\partial u_z}{\partial z}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad e_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \quad e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

The vanishing of the divergence of the stress tensor may be written

$$0 = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}, \quad 0 = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}, \quad 0 = \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

The corresponding formulae for cylindrical (ϖ, ϕ, z) coordinates are

$$e_{\varpi\varpi} = \frac{\partial u_\varpi}{\partial \varpi}, \quad e_{\phi\phi} = \frac{1}{\varpi}\frac{\partial u_\phi}{\partial \phi}, \quad e_{zz} = \frac{\partial u_z}{\partial z}$$

$$e_{\varpi z} = \frac{1}{2}\left(\frac{\partial u_z}{\partial \varpi} + \frac{\partial u_\varpi}{\partial z}\right), \quad e_{z\phi} = \frac{1}{2}\left(\frac{\partial u_\phi}{\partial z} + \frac{1}{\varpi}\frac{\partial u_z}{\partial \phi}\right), \quad e_{\varpi\phi} = \frac{1}{2}\left(\frac{1}{\varpi}\frac{\partial u_\varpi}{\partial \phi} + \frac{\partial u_\phi}{\partial \omega} - \frac{u_\phi}{\varpi}\right)$$

$$0 = \frac{1}{\varpi}\frac{\partial(\varpi\sigma_{\varpi\varpi})}{\partial \varpi} + \frac{1}{\varpi}\frac{\partial \sigma_{\varpi\phi}}{\partial \phi} + \frac{\partial \sigma_{\varpi z}}{\partial z} - \frac{\sigma_{\phi\phi}}{\varpi}$$

$$0 = \frac{1}{\varpi^2}\frac{\partial(\varpi^2\sigma_{\varpi\phi})}{\partial \varpi} + \frac{1}{\varpi}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\partial \sigma_{\phi z}}{\partial z}$$

$$0 = \frac{1}{\varpi}\frac{\partial(\varpi\sigma_{z\varpi})}{\partial \varpi} + \frac{1}{\varpi}\frac{\partial \sigma_{z\phi}}{\partial \phi} + \frac{\partial \sigma_{zz}}{\partial z}$$

and for spherical (r, θ, ϕ) coordinates

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot\theta}{r}u_\theta$$

$$e_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{\cot\theta}{r}u_\phi\right), \quad e_{r\phi} = \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r}\right), \quad e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$0 = \frac{1}{r^2}\frac{\partial(r^2\sigma_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\sigma_{r\theta})}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\sigma_{r\phi})}{\partial \phi} - \frac{(\sigma_{\theta\theta} + \sigma_{\phi\phi})}{r}$$

$$0 = \frac{1}{r^3}\frac{\partial(r^3\sigma_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\sigma_{\theta\theta})}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\sigma_{\theta\phi})}{\partial \phi} - \frac{\cot\theta\,\sigma_{\phi\phi}}{r}$$

$$0 = \frac{1}{r^3}\frac{\partial(r^3\sigma_{r\phi})}{\partial r} + \frac{1}{r\sin^2\theta}\frac{\partial(\sin^2\theta\,\sigma_{\theta\phi})}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\sigma_{\phi\phi})}{\partial \phi}$$

The following formulae relate the Lamé constants and the more familiar engineering constants — the shear modulus, *G*, Young's modulus, *E*, and Poisson's ratio, ν:

$$E = \frac{\mu\left(3\lambda + 2\mu\right)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}; \quad \mu = G = \frac{E}{2(1 + \nu)}, \quad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}$$

Note that ν —> 1/2 has the corresponding limit λ —> ∞. The material in either case is incompressible.

References

Little, R. W. *Elasticity* Prentice-Hall: Englewood Cliffs, NJ 1973

McConnell, A. J. *Applications of Tensor Analysis* Dover: NY 1957