A Slight Variation on Glicksberg's Theorem

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Glicksberg's (1952) fixed point theorem is stated for a correspondence defined on a convex, compact subset of a locally convex Hausdorff linear space. One of the main applications of this result, the existence of mixed strategy equilibria in compact, continuous games, focuses on a metrizable subset of the vector space of signed measures, endowed with the weak* topology. I thought it would be interesting to re-prove the fixed point theorem imposing restrictions only on the metric subspace, rather than the vector space in which it is imbedded. The result is a statement of Glicksberg's fixed point theorem that doesn't require knowledge of topological vector spaces. It is enough to understand the basics of vector spaces and metric spaces.

Assume V is a vector space, and let X be a convex subset endowed with a metric d. The key condition on this metric is that d is *compatible* with V, in the sense that:

(i) $x^n \to x$ and $y^n \to y$ in X and $\alpha^n \to \alpha$ in $[0,1]$ implies $\alpha^n x^n + (1-\alpha^n)y^n \to \alpha x + (1-\alpha)y$.

(ii) for all $x, y, z, w \in X$, all $\alpha \in [0,1]$, and all $\beta \in \Re$, $d(x,y) \leq \beta$ and $d(w,z) \leq \beta$ implies $d(\alpha x + (1-\alpha)w, \alpha y + (1-\alpha)z) \leq \beta$.

I use the usual notation $B_r(x)$ for an open ball of radius r around x.

A SLIGHT VARIATION ON GLICKSBERG'S THEOREM Let V be a vector space, let $X \subseteq V$ be convex, and let d be a metric on X such that X is compact and d is compatible with V. If $\varphi\colon X \rightarrow\rightarrow X$ has nonempty, convex values and has closed graph, then there exists $x^* \in X$ such that $x^* \in \varphi(x^*)$.

The proof of the theorem proceeds in a few steps.

Finite approximation. For each natural number m, $\{B_{1/m}(y) \mid y \in X\}$ is an open cover of X, so it has a finite cover consisting of the open balls of radius $1/m$ centered at points $x_1(m), \ldots, x_{k(m)}(m) \in X$. Let $Y(m) = \text{conv}\{x_1(m), \ldots, x_{k(m)}(m)\}$.

Constructing a homeomorphism. Let $Z(m) = \text{span}\{x_1(m), \ldots, x_{k(m)}(m)\}$, and let $\{z_1(m), \ldots, z_{\ell(m)}(m)\} \subseteq \{x_1(m), \ldots, x_{k(m)}(m)\}$ be a basis of $Z(m)$. Define the mapping

$f(m)\colon Z(m) \to \Re^{\ell(m)}$ by $f(m)(y) = (\alpha_1, \ldots, \alpha_{\ell(m)})$, where $y = \sum_{i=1}^{\ell(m)} \alpha_i z_i(m)$. Thus, $f(m)$ is linear, 1-1, and onto. Let $g(m)$ denote the restriction of $f(m)$ to $Y(m)$, and let $g(m)_i$ denote the ith coordinate mapping corresponding to $g(m)$. Thus, for each $y \in Y(m)$, we have $y = \sum_{i=1}^{\ell(m)} g(m)_i(y) z_i(m)$. Note that $g(m)(Y(m)) = E(m)$, where $E(m) \subseteq \Re^{\ell(m)}$ is defined by

$$E(m) \;=\; \mathrm{conv}\{f(m)(x_1(m)), \ldots, f(m)(x_{k(m)}(m))\}.$$

To see that $g(m)\colon Y(m) \to E(m)$ is continuous, suppose $y^n \to y$ in $Y(m)$. For each n, we have $y^n = \sum_{i=1}^{\ell(m)} g(m)_i(y^n) z_i(m)$ and $y = \sum_{i=1}^{\ell(m)} g(m)_i(y) z_i(m)$. By (i), we have $(1/2)y^n + (1/2)y \to y$. Suppose $g(m)_j(y^n) \not\to g(m)_j(y)$ for some coordinate j. Then $(1/2)g(m)_j(y^n) + (1/2)g(m)_j(y) \not\to g(m)_j(y)$, and then

$$\begin{aligned}
&(1/2)y^n + (1/2)y \\
&\quad = (1/2)\left(\sum_{i=1}^{\ell(m)} g(m)_i(y^n) z_i(m)\right) + (1/2)\left(\sum_{i=1}^{\ell(m)} g(m)_i(y) z_i(m)\right) \\
&\quad = \sum_{i=1}^{\ell(m)} [(1/2)g(m)_i(y^n) + (1/2)g(m)_i(y)]\, z_i(m) \\
&\quad \not\to \sum_{i=1}^{\ell(m)} g(m)_i(y) z_i(m) \\
&\quad = y,
\end{aligned}$$

a contradiction. Therefore, $g(m)(y^n) \to g(m)(y)$. To see that $g(m)^{-1}\colon E(m) \to Y(m)$ is continuous, suppose $\alpha^n \to \alpha$ in $E(m)$. Then

$$g(m)^{-1}(\alpha^n) \;=\; \sum_{i=1}^{\ell(m)} \alpha_i^n z_i(m) \;\to\; \sum_{i=1}^{\ell(m)} \alpha_i z_i(m) \;=\; g(m)^{-1}(\alpha)$$

follows from (i).

Constructing another correspondence. Define $\varphi(m)\colon Y(m) \to\!\!\!\to Y(m)$ by

$$\varphi(m)(x) \;=\; \mathrm{conv}\{x_i(m) \mid \exists w \in \varphi(x) \text{ s.t. } d(x_i(m), w) = \min_{j=1,\ldots,k(m)} d(x_j(m), w)\}.$$

Since φ has nonempty values and the minimization above is over a finite set, $\varphi(m)$ has nonempty values. It is clear that $\varphi(m)$ has convex values. To see that it has closed graph, suppose $x^n \to x$ and $y^n \to y$ in X and $y^n \in \varphi(m)(x^n)$ for all n. For each n, there exists $\beta^n \in \Re_+^{k(m)}$ such that $\sum_{i=1}^{k(m)} \beta_i^n = 1$, that $y^n = \sum_{i=1}^{k(m)} \beta_i^n x_i(m)$, and that $\beta_i^n > 0$ implies there exists $w_i^n \in \varphi(x^n)$ such that $d(x_i(m), w_i^n) = \min\{d(x_j(m), w_i^n) \mid j = 1, \ldots, k(m)\}$. We may suppose, by compactness of the unit simplex, that $\beta^n \to \beta$. By linearity of $g(m)$, we have

$$g(m)(y^n) \;=\; \sum_{i=1}^{k(m)} \beta_i^n g(m)(x_i(m)) \;\to\; \sum_{i=1}^{k(m)} \beta_i g(m)(x_i(m)) \;=\; g(m)(\sum_{i=1}^{k(m)} \beta_i x_i(m)).$$

Therefore, $y = \sum_{i=1}^{k(m)} \beta_i x_i(m)$. Take any i such that $\beta_i > 0$, implying $\beta_i^n > 0$ for high enough n, implying the existence of $w_i^n \in \varphi(x^n)$ such that $d(x_i(m), w_i^n) = \min\{d(x_j(m), w_i^n) \mid j = 1, \ldots, k(m)\}$ for high enough n. By compactness, we may suppose $w_i^n \to w_i$. Since φ has closed graph, we have $w_i \in \varphi(x)$. Furthermore, $d(x_i(m), w_i) = \min\{d(x_j(m), w_i) \mid j = 1, \ldots, k(m)\}$. Therefore, $y \in \varphi(m)(x)$, as required.

Finding a fixed point. Define the correspondence $\Phi(m)\colon E(m) \to\!\!\to E(m)$ by

$$\Phi(m)(\alpha) \quad = \quad g(m)(\varphi(m)(g(m)^{-1}(\alpha))).$$

Since $g(m)$ is a linear homeomorphism, this defines a correspondence on a compact, convex subset of $\Re^{\ell(m)}$ that has nonempty, convex values and has closed graph. Therefore, there exists $\alpha^*(m) \in E(m)$ such that $\alpha^*(m) \in \Phi(m)(\alpha^*(m))$. Define $x^*(m) = g(m)^{-1}(\alpha^*(m))$, and note that $x^*(m) \in \varphi(m)(x^*(m))$. For each m, there exist $\beta(m) \in \Re_+^{k(m)}$ such that $\sum_{i=1}^{k(m)} \beta_i(m) = 1$, that $x^*(m) = \sum_{i=1}^{k(m)} \beta_i(m) x_i(m)$, and that $\beta_i(m) > 0$ implies $d(x_i(m), w_i(m)) = \min\{d(x_j(m), w_i(m)) \mid j = 1, \ldots, k(m)\}$ for some $w_i(m) \in \varphi(x^*(m))$. Thus, $d(x_i(m), w_i(m)) \leq 1/m$. By (ii), we have $d(x^*(m), w^*(m)) \leq 1/m$, where $w^*(m) = \sum_{i=1}^{k(m)} \beta_i(m) w_i(m)$. Since $\varphi(x^*(m))$ is convex, we have $w^*(m) \in \varphi(x^*(m))$. By compactness, we may suppose that $x^*(m) \to x^*$ and $w^*(m) \to w^*$, and from the foregoing it follows that $x^* = w^*$. Since φ has closed graph, $x^* = w^* \in \varphi(x^*)$, completing the proof.

PURE STRATEGY EQUILIBRIUM EXISTENCE The above Theorem can be used to prove, using standard arguments, that compact, quasi-concave, continuous strategic form games have pure strategy equilibria. Basically, we let each of n players have a set X_i of pure strategies, assumed to lie in some ambient vector space V_i and to be endowed with a metric d_i compatible with V_i. Obviously, the main application of interest is when X_i is a subset of finite-dimensional Euclidean space. We let $X = \prod X_i$ denote the set of pure strategy profiles, endowed with the maximum of the individual metrics, i.e., $d(x, y) = \max\{d(x_i, y_i) \mid i = 1, \ldots, n\}$, and we imbed X in the product of vector spaces $V = \prod V_i$. It is easy to see that d is compatible with V. Each player i has a payoff function $u_i\colon X \to \Re$. If each X_i is compact and if each u_i is continuous in $(x_1, \ldots, x_n)$ and quasi-concave in x_i, then the game $((X_i, u_i), i = 1, \ldots, n)$ has a pure strategy Nash equilibrium. To prove this, we just define the product of best response correspondences. Our convexity assumptions ensure that this is convex-valued, while our continuity assumptions ensure that it has nonempty values and closed graph. Thus, it has a fixed point, which yields the desired equilibrium point.

MIXED STRATEGY EQUILIBRIUM EXISTENCE If we drop the convexity assumptions from the above description of a strategic form game, pure strategy equilibria need not exist. Suppose that each X_i is a separable metric space, and let $\mathcal{P}(X_i)$ denote the set of Borel probability measures on X_i, denoted σ, μ, ν, etc. We view $\mathcal{P}(X_i)$ as a subset of the vector space of signed measures on X_i, say V_i, and we endow this subset with the Prohorov metric ρ_i, defined by

$$\rho_i(\mu, \nu) \quad = \quad \inf\{\epsilon > 0 \mid \text{ for all Borel } B, \mu(B) \leq \nu(B^\epsilon) + \epsilon \text{ and } \nu(B) \leq \mu(B^\epsilon) + \epsilon\},$$

where $B^\epsilon = \{x \in X \mid d(x, B) < \epsilon\}$. Note that, if $\rho_i(\mu^n, \mu) \leq \epsilon$ and $\rho_i(\nu^n, \nu) \leq \epsilon$, then, for all Borel B and all $\alpha \in [0, 1]$,

$$\alpha\mu^n(B) + (1-\alpha)\nu^n(B) \quad \leq \quad \alpha\mu(B^\epsilon) + (1-\alpha)\nu(B^\epsilon) + \epsilon,$$

and visa versa. Thus, for all $\alpha \in [0, 1]$, we have $\rho_i(\alpha\mu^n + (1-\alpha)\nu^n, \alpha\mu + (1-\alpha)\nu) \leq \epsilon$. Similarly, if $|\alpha^n - \alpha| \leq \epsilon$, then, for all $\mu, \nu \in \mathcal{P}(X_i)$, we have $\rho_i(\alpha^n\mu + (1-\alpha^n)\nu, \alpha\mu + (1-\alpha)\nu) \leq \epsilon$. Therefore, if $\mu^n \to \mu$, if $\nu^n \to \nu$, and if $\alpha^n \to \alpha$, then, for every $\epsilon > 0$, we can choose n high enough that

$$\begin{aligned}
&\rho_i(\alpha^n\mu^n + (1-\alpha^n)\nu^n, \alpha\mu + (1-\alpha)\nu) \\
&\quad \leq \quad \rho_i(\alpha^n\mu^n + (1-\alpha^n)\nu^n, \alpha^n\mu + (1-\alpha^n)\nu) \\
&\qquad\quad + \rho_i(\alpha^n\mu + (1-\alpha^n)\nu, \alpha\mu + (1-\alpha)\nu) \\
&\quad \leq \quad (\epsilon/2) + (\epsilon/2) \\
&\quad = \quad \epsilon,
\end{aligned}$$

and we conclude that $\alpha^n\mu^n + (1-\alpha^n)\nu^n \to \alpha\mu + (1-\alpha)\nu$, as in (i). For all $\mu, \nu, \mu', \nu' \in \mathcal{P}(X_i)$, all $\alpha \in [0, 1]$, and all $\beta \in \Re$, if

$$\mu(B) \leq \nu(B^\beta) + \beta \quad \text{and} \quad \mu'(B) \leq \nu'(B^\beta) + \beta,$$

(and visa versa) for all Borel B, then we have

$$\alpha\mu(B) + (1-\alpha)\mu'(B) \quad \leq \quad \alpha\nu(B^\beta) + (1-\alpha)\nu'(B^\beta) + \beta$$

(and visa versa), as required for (ii). Thus, ρ_i is compatible with V_i. We then endow $\prod \mathcal{P}(X_i)$ with the maximum of the individual metrics, say ρ, and we imbed it in the product of vector spaces $\prod V_i$. As above, ρ will be compatible with $\prod V_i$. Assuming each X_i is a compact metric space, then, because the topology induced by ρ_i is the weak* topology, each

$\mathcal{P}(X_i)$ is compact, and so is $\prod \mathcal{P}(X_i)$. Finally, we extend player i's payoff function on pure strategy profiles to mixed strategy profiles $(\sigma_1, \dots, \sigma_n)$ as

$$U_i(\sigma_1, \dots, \sigma_n) = \int u_i d\sigma,$$

where $\sigma = \sigma_1 \times \cdots \times \sigma_n$ is the product measure. Of course U_i is multi-linear in mixed strategies and, therefore, quasi-concave in σ_i. If u_i is continuous, then so is U_i. Thus, we have defined a strategic form game $((\mathcal{P}(X_i), U_i), i = 1, \dots, n)$, called the *mixed extension* of $((X_i, u_i), i = 1, \dots, n)$, satisfyin the conditions for existence of a pure strategy equilibrium. We conclude that the mixed extension has a pure strategy equilibrium, which corresponds to a mixed strategy equilibrium of the original game.

References

[1] I. Glicksberg (1952) "A Further Generalization of the Kakutani Fixed Point Theorem, with Application to Nash Equilibrium Points," *Proceedings of the American Mathematical Society,* 3:170-174.