Technical Notes

Converting Erdélyi [1] to Bracewell [2, 3]

Erdélyi defined his Hankel transform of order v as [1]:

$$g(y;v) = \int_0^\infty f(x) J_v(xy)(xy)^{1/2}\, dx$$

In this project, we just consider all the Hankel transforms of order 0. We unify the Erdélyi version into Bracewell's version [2, 3], which is more easily related to standard Fourier transform notation.

- For f(x), we divided the functions by $x^{1/2}$, and change x to $2\pi r$.
- For the Hankel transform, we divided the functions by $2\pi y^{1/2}$, and then change y to q directly.

Classification

For each function we made, we also marked its classification for the following criteria as true or false:

1) Oscillating
2) Finite domain
3) Singularity
4) Non negative
5) Monotonic
6) Eigen function

You can use our search engine to find the functions which fit your own criteria.

Notes

1) The range we chose for graphing each function is $r \text{ or } q = 2\times 5^n, n\in Z$, based on the extent of each function.

For example, when $n=-1$, $r = 2\times 5^{-1} = 0.4$; when $n=0$, $r = 2\times 5^0 = 2$

So in our project, you can only find the range equal to 0.4, 2, 10, and 50 (maximum).

2) All the functions were made using Mathematica (Wolfram Research, Champaign, IL, USA), and for plotting purposes the Dirac Delta function is given a width of 0.2 and height of 1000.
3) All the functions are verified using numerical integration except for a few which are impossible to verify by numerical integration (for example the Dirac Delta function (bracewell_pg249_3)).
4) Π function's definition

$$\Pi = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & |x| \geq \frac{1}{2} \end{cases}$$

5) Orthogonal polynomials

Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

Jacobi polynomial

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left[(1-x)^{n+\alpha} (1+x)^{n+\beta}\right]$$

Laguerre polynomial

$$L_n^{\alpha}(z) = \frac{e^z z^{-\alpha}}{n!} \frac{d^n}{dz^n} \left(e^{-z} z^{n+\alpha}\right)$$

$$L_n^0(z) = L_n(z)$$

6) Legendre functions

$$P_v^{\mu}(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} {}_2F_1\left(-v, v+1; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2} z\right)$$

$$Q_v^{\mu}(z) = \frac{e^{\mu\pi i} \pi^{\frac{1}{2}} \Gamma(\mu+v+1)}{2^{v+1} \Gamma(v+3/2)} z^{-\mu-v-1} \left(z^2-1\right)^{\frac{1}{2}\mu} \times {}_2F_1\left(\frac{\mu+v+1}{2}, \frac{\mu+v+2}{2}; v+\frac{3}{2}; \frac{1}{z^2}\right)$$

z in the complex plane cut along the real axis from -1 to 1.

$$P_v^{\mu}(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu} {}_2F_1\left(-v, v+1; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2} x\right), -1 < x < 1$$

$$Q_v^{\mu}(x) = \frac{1}{2} e^{-i\mu\pi} \left[e^{-\frac{1}{2} i\mu\pi} Q_v^{\mu}(x+i0) + e^{\frac{1}{2} i\mu\pi} Q_v^{\mu}(x-i0)\right], -1 < x < 1$$

$$P_v(z) = P_v^0(z), Q_v(z) = Q_v^0(z).$$

7) Bessel functions and related functions

$$J_v(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2} z\right)^{v+2m}}{m! \Gamma(v+m+1)}$$

$$Y_v(z) = \operatorname{cosec} v\pi \left[J_v(z) \cos v\pi - J_{-v}(z)\right]$$

$$H_v^{(1)}(z) = J_v(z) + iY_v(z)$$

$$H_v^{(2)}(z) = J_v(z) - iY_v(z)$$

Modified Bessel functions

$$I_v(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{v+2m}}{m!\Gamma(v+m+1)}$$

$$K_v(z) = \frac{\pi}{2}\frac{I_{-v}(z) - I_v(z)}{\sin v\pi}$$

Kelvin's and related functions

$$\mathrm{ber}_v(z) + i\,\mathrm{bei}_v(z) = J_v\left(ze^{1/4\pi i}\right)$$

$$\mathrm{ber}_v(z) - i\,\mathrm{bei}_v(z) = J_v\left(ze^{-1/4\pi i}\right)$$

$$\mathrm{ker}_v(z) + i\,\mathrm{kei}_v(z) = K_v\left(ze^{1/4\pi i}\right)$$

$$\mathrm{ker}_v(z) - i\,\mathrm{kei}_v(z) = K_v\left(ze^{-1/4\pi i}\right)$$

$$\mathrm{ber}(z) = \mathrm{ber}_0(z), \mathrm{bei}(z) = \mathrm{bei}_0(z)$$

$$\mathrm{ker}(z) = \mathrm{ker}_0(z), \mathrm{kei}(z) = \mathrm{kei}_0(z)$$

Struve's functions

$$\begin{aligned}\mathrm{H}_v(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m \left(z/2\right)^{v+2m+1}}{\Gamma\left(m+3/2\right)\Gamma\left(v+m+3/2\right)} \\ &= \frac{\left(z/2\right)^{v+1}}{\Gamma\left(3/2\right)\Gamma\left(v+3/2\right)}\, {}_1F_2\left(1; 3/2, v+3/2; -z^2/4\right) \\ &= 2^{1-v}\pi^{-1/2}\left[\Gamma\left(v+1/2\right)\right]^{-1} s_{v,v}(z)\end{aligned}$$

8) Hypergeometric functions

$${}_mF_n(a_1,\ldots,a_m;\gamma_1,\ldots,\gamma_n;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_m)_k}{(\gamma_1)_k \ldots (\gamma_n)_k} \frac{z^k}{k!}$$

${}_2F_1(a,b;c;z)$ is Gauss'denoted by hypergeometric series and is often denoted by $F(a,b;c;z)$.

$_1F_1(a,b;c;z)$ is Kummer's confluent hypergeometric series and is sometimes denoted by $\Phi(a;c;z)$.

9) Confluent hypergeometric functions

Whittaker's function

$$M_{k,\mu}(z)=z^{1/2+\mu}e^{-1/2z}\,{}_1F_1\left(1/2+\mu-k;2\mu+1;z\right)$$

$$W_{k,\mu}(z)=\frac{\Gamma(-2\mu)M_{k,\mu}(z)}{\Gamma\left(1/2-\mu-k\right)}+\frac{\Gamma(2\mu)M_{k,-\mu}(z)}{\Gamma\left(1/2+\mu-k\right)}$$

Parabolic cylinder functions

$$D_v(z)=2^{1/2v+1/4}z^{-1/2}W_{1/2v+1/4,1/4}\left(1/2z^2\right)$$

$$D_n(z)=(-1)^n e^{1/4z^2}\frac{d^n}{dz^n}\left(e^{-1/2z^2}\right)$$

$$\mathrm{si}(x)=-\int_x^\infty\frac{\sin t}{t}dt=\frac{1}{2i}\left[\mathrm{Ei}(ix)-\mathrm{Ei}(-ix)\right]$$

$$\mathrm{Si}(x)=\int_x^\infty\frac{\sin t}{t}dt=1/2\pi+\mathrm{si}(x)$$

$$\mathrm{Ci}(x)=-\int_x^\infty\frac{\cos t}{t}dt=-\mathrm{ci}(x)=\frac{1}{2i}\left[\mathrm{Ei}(ix)+\mathrm{Ei}(-ix)\right]$$

10) Elliptic functions and integrals

$$K(k)=\int_0^{1/2\pi}\left(1-k^2\sin^2\phi\right)^{-1/2}d\phi=1/2\pi\,{}_2F_1\left(1/2,1/2;1;k^2\right)$$

$$E(k)=\int_0^{1/2\pi}\left(1-k^2\sin^2\phi\right)^{1/2}d\phi=1/2\pi\,{}_2F_1\left(-1/2,1/2;1;k^2\right)$$

Errors/Typos in books

1) erdelyi_8.2_17: Added to the 2μ power to the end, this has been verified by numerical integration.
2) erdelyi_8.2_38: Changed a^2 to a. There is no reason for the squared there. Added a>0 constraint to compensate.
3) erdelyi_8.2_39: Changed a^2 to a. There is no reason for the squared there. Added a>0 constraint to compensate.

   Added a negative sign to $F(q)$, numerical integration showed that it was missing the negative sign

4) erdelyi_8.2_46: b>0 is required, numerical integration fail for b=-1, (a<0 is fine since all a term is square of symmetric)

5) erdelyi_8.2_47: Changed a^2 to a. There is no reason for the squared there. Added a>0 constraint to compensate.

6) erdelyi_8.3_4: Replace x^2 in Hankel transform function to y^2.

7) erdelyi_8.3_7: Replace x^2 in Hankel transform function to y^2.

8) erdelyi_8.3_43: For $F(q)$, multiply factor 2, verified by numerical integration.

9) Bracewell Page335 Fig. 13.3 replace $\prod(v)$ by $\sqrt{\frac{\pi}{2}}\delta(\pi-v)+\sqrt{\frac{\pi}{2}}\delta(\pi+v)$.

References

1. Erdélyi, A. and H. Bateman, *Tables of integral transforms. vol. 2*. 1954, McGraw-Hill: New York. 451.
2. Bracewell, R.N., *The Fourier transform and its applications*. 3rd ed. 2000, Boston: WCB/McGraw-Hill.
3. Andrews, L.C., B.K. Shivamoggi, and Society of Photo-optical Instrumentation Engineers., *Integral transforms for engineers*. 1999, Bellingham, Wash.: SPIE Optical Engineering Press. x, 353 p.
4. Andrews, L.C. and R.L. Phillips, *Mathematical techniques for engineers and scientists*. 2003, Bellingham: SPIE-The International Society for Optical Engineering. XV, 797 p.
5. Poularikas, A.D., *The handbook of formulas and tables for signal processing*. The electrical engineering handbook series. 1999, Boca Raton, Fla. u.a.: CRC Press u.a. Getr. Zählung [ca. 800 S.].
6. Parker, K.J., *Lecture Notes, Medical Imaging: Theory and Implementation*. 2013, University of Rochester.
7. Bracewell, R.N., *Two-dimensional imaging*. Prentice-Hall signal processing series. 1995, Englewood Cliffs, N.J.: Prentice Hall. xiv, 689 p.
8. Wikipedia. *Fourier transform*. 9 August 2013 21:20 UTC Available from: http://en.wikipedia.org/w/index.php?title=Fourier_transform&oldid=567869073.
9. Wikipedia. *Hankel transform* 9 August 2013 12:02 UTC Available from: http://en.wikipedia.org/w/index.php?title=Hankel_transform&oldid=567808780.
10. Waag, R.C., J.A. Campbell, J. Ridder, and P.R. Mesdag, *Cross-sectional Measurements and Exploitations of ultrasonic fields. 1985,* IEEE Transactions on Sonics and Ultrasonics 32(1)26-35.

Acknowledgements

We are grateful to the Hajim School of Engineering and Applied Sciences and the Wadsworth C. Sykes Engineering Faculty Award for supporting this work. The excellent work of University of Rochester students Shuopeng Deng and Yuchuan Zhuang is gratefully noted. We are also indebted to Professor Al Clark for his patient advice.