Tutorial

Many of the linear transforms in common use have a direct connection with either the Fourier or the Laplace transform [1-7]. The closest relationship is with the generalizations of the Fourier transform [8] to two or more dimensions, and with the Hankel transforms [9] of the zero and higher orders, into which the multidimensional Fourier transforms degenerate under circumstances of symmetry.

The two-dimensional Fourier transform

The variable x may stand for some physical quantity such as time or frequency, which is essentially one-dimensional, or it may be the coordinate in a one-dimensional physical system such as a stretched string or an electrical transmission line. However, in cases which are two dimensional-stretched membranes, antennas and arrays of antennas, lenses and diffraction gratings, pictures on television screens, and so on, more general formulas apply.

A two-dimensional function $f(x,y)$ has a two-dimensional transform $F(u,v)$, and between the two the following relations exist:

$$F(u,v)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-i2\pi(ux+vy)}dxdy$$
$$f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)e^{i2\pi(ux+vy)}dudv$$

These equations describe an analysis of the two-dimensional function $f(x,y)$ into components of the form $\exp[i2\pi(ux+vy)]$. Since any such component can be split into cosine and sine parts, we may begin by considering a cosine component $\cos[2\pi(ux+vy)]$.

The magnitude of $F(u_0,v_0)$ tells us how much $f(x,y)$ looks like a cosine function heading in the direction $\theta=\arctan\left(v_0/u_0\right)$. For example, if $f(x,y)=\cos[2\pi(x\cos\theta+y\sin\theta)]$, then $F(u,v)$ is a pair of impulses located at an angle θ with respect to the u-axis (see Figure 1).

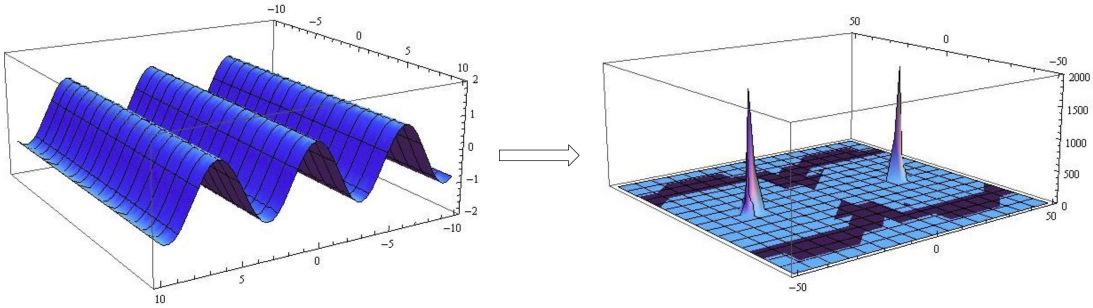


Figure 1 2-Dimensional Fourier transform pair

As a second example of a two-dimensional function, consider the height of the ground at the geographical point (x, y), over the area occupied by the mountain which is conventionally represented in Figure 2 by contours of constant height. The function $\cos[2\pi(u_0 x + v_0 y)]$ represents a consinusoidally corrugated land surface whose contours of constant height coincide with lines whose equation is

$$u_0 x + v_0 y = const$$

The corrugations face in a direction that makes an angle $\arctan(v_0/u_0)$ with the x axis and their wavelength is $\left(u_0^2 + v_0^2\right)^{-\frac{1}{2}}$. If a section is made through the corrugations, in the x direction, it will undulate with a frequency of u_0 cycles per unit of x. Similarly, v_0 may be interpreted as the number of cycles per unit of u_0, in the y direction.

In Figure 2, a prominent Fourier component of the mountain is shown. In the transform domain the complex component is characterized in wavelength and orientation by the point (u, v) in the $u - v$ plane and its amplitude by $F(u, v)$. The interpretation of u and v as spatial frequencies is emphasized by dimensioning u^{-1} and v^{-1}, the wavelengths of sections taken in the x and y direction, respectively (see Figure 2). The second of the Fourier relations quoted above asserts that a summation of corrugations of appropriate wavelengths and orientations, taken with suitable amplitudes, can reproduce the original mountain. The sinusoidal components, which must also be included, allow for the possibility that the corrugations may have to be slid into appropriate spatial phases.

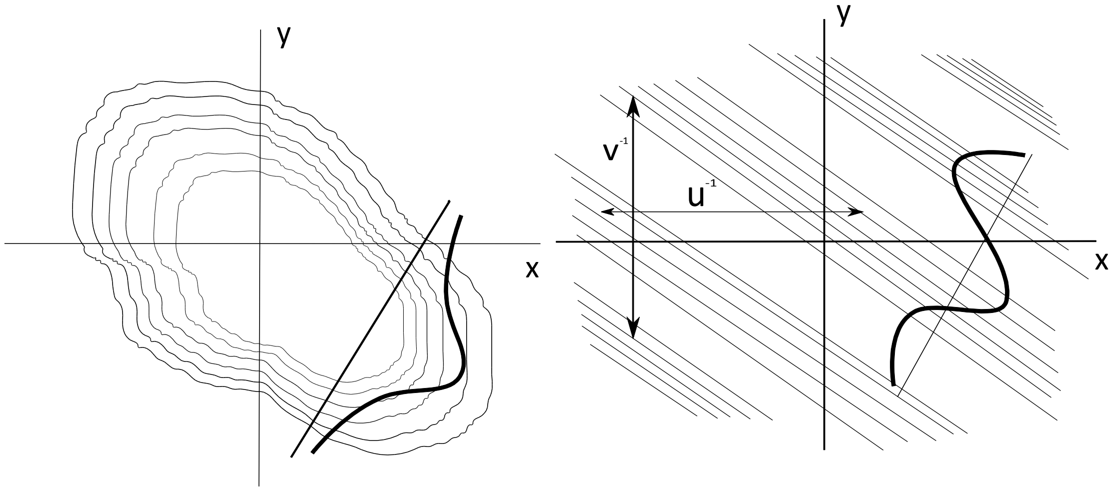


Figure 2　A mountain（left）and a prominent Fourier component thereof (right)

Schwarz's Inequality

Another way to think about the meaning of $F(u_0, v_0)$ as a large or small number compared to a sinusoid, is to consider Schwarz's Inequality. Let $\psi_1(x)$ and $\psi_2(x)$ be any two real integrable function in $\left[a,b\right]$, then Schwarz's Inequality is given by

$$\left[\int_a^b \psi_1(x)\psi_2(x)dx\right]^2 \leq \int_a^b \left[\psi_1(x)\right]^2 dx \int_a^b \left[\psi_2(x)\right]^2 dx$$

When equality iff* $\psi_1(x) = \alpha\psi_2(x)$ with α a constant. Thus, if $\psi_1 = f\left(x, y\right)$ and $\psi_2 = \mathbf{e}^{-i2\pi\left(u_0 x + v_0 y\right)}$, then $\left|F\left(u_0, v_0\right)\right|^2$ reaches a maximum when $f\left(x, y\right)$ approaches a sinusoid with wavelengths and angle matching $\left(u_0, v_0\right)$. Schwarz's Inequality is also called the Cauchy-Schwarz inequality or Buniakowsky inequality.

- iff* : If and only if (i.e., necessary and sufficient). The terms "just if" or "exactly when" are sometimes used instead.

Theorems for the two-dimensional Fourier transform [2]

Theorem	$f(x,y)$	$F(u,v)$
Similarity	$f(ax,by)$	$\frac{1}{\lvert ab\rvert}F\left(\frac{u}{a},\frac{v}{b}\right)$
Addition	$f(x,y)+g(x,y)$	$F(u,v)+G(u,v)$
Shift	$f(x-a,y-b)$	$e^{-2\pi i(au+bv)}F(u,v)$
Modulation	$f(x,y)\cos\omega x$	$\frac{1}{2}F\left(u+\frac{\omega}{2\pi},v\right)+\frac{1}{2}F\left(u-\frac{\omega}{2\pi},v\right)$
Convolution	$f(x,y)*g(x,y)$	$F(u,v)G(u,v)$
Autocorrelation	$f(x,y)*f^{*}(-x,-y)$	$\lvert F(u,v)\rvert^2$
Rayleigh	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\lvert f(x,y)\rvert^2\,dxdy=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\lvert F(u,v)\rvert^2\,dudv$	
Power	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)*g^{*}(x,y)dxdy=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)*G^{*}(u,v)dudv$	
Parseval	$\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\lvert f(x,y)\rvert^2=\sum\sum a_{mn}^2,$ Where $F(u,v)=\sum\sum a_{mn}[{}^2\delta(u-m,v-n)]$	
Differentiation	$\left(\frac{\partial}{\partial x}\right)^m\left(\frac{\partial}{\partial y}\right)^n f(x,y)$	$(2\pi iu)^m(2\pi iv)^n F(u,v)$
	$\frac{\partial}{\partial x}f(x,y)$	$2\pi iuF(u,v)$
	$\frac{\partial}{\partial y}f(x,y)$	$2\pi ivF(u,v)$
	$\frac{\partial^2}{\partial x^2}f(x,y)$	$-4\pi^2u^2F(u,v)$
	$\frac{\partial^2}{\partial y^2}f(x,y)$	$-4\pi^2v^2F(u,v)$
	$\frac{\partial^2}{\partial x\partial y}f(x,y)$	$-4\pi^2uvF(u,v)$

	$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f(x,y)$	$-4\pi^2(u^2+v^2)F(u,v)$
Definite integral	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = F(0,0)$	
First moments	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xf(x,y)dxdy = \frac{1}{-2\pi i}F'_u(0,0)$	
	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x\cos\theta+y\sin\theta)f(x,y)dxdy = \frac{1}{-2\pi i}\left[\cos\theta F'_u(0,0)+\sin F'_v(0,0)\right]$	
Center of gravity	$\langle x\rangle = \frac{F'_u(0,0)}{-2\pi i F(0,0)}$	$\langle y\rangle = \frac{F'_v(0,0)}{-2\pi i F(0,0)}$
Second moments	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,y)dxdy = \frac{F''_{uu}(0,0)}{-4\pi^2}$	
	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf(x,y)dxdy = \frac{F''_{uv}(0,0)}{-4\pi^2}$	
	$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x^2+y^2)f(x,y)dxdy = \frac{1}{-4\pi^2}\left[F''_{uu}(0,0)+F''_{vv}(0,0)\right]$	
Equivalent width	$\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy}{f(0,0)} = \frac{F(0,0)}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)dudv}$	
Finite differences	$\Delta_x f(x,y)$	$i2\sin\pi u F(u,v)$
	$\Delta_{xy}^2 f(x,y)$	$-4\sin\pi u\sin\pi v F(u,v)$
	$\Delta_{xx}^2 f(x,y)$	$-4(\sin\pi u)^2 F(u,v)$
Running means	$\left[\Pi\left(\frac{x}{a}\right)\Pi\left(\frac{y}{b}\right)\right]*f(x,y)$	$ab\sin cau\sin cbvF(u,v)$
Separable product	$f(x)g(y)$	$F(u)G(v)$

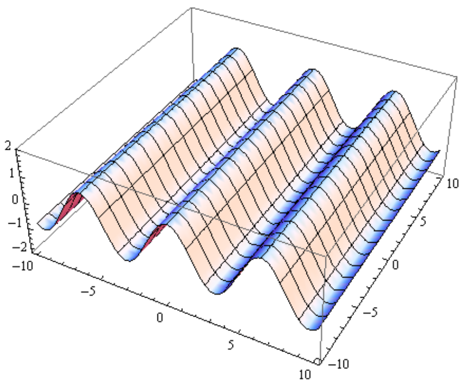
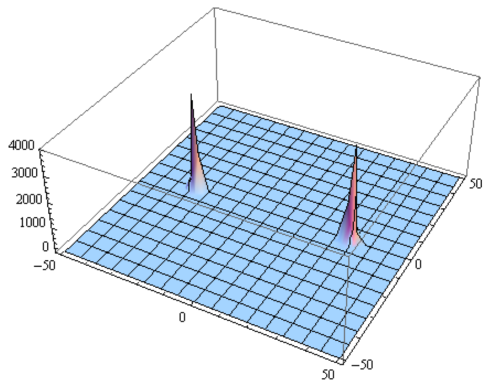
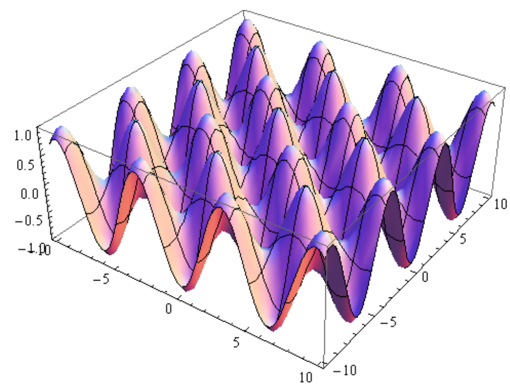
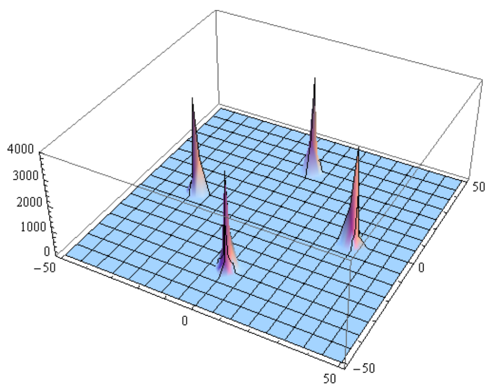
The finite differences in the table are defined as follows:

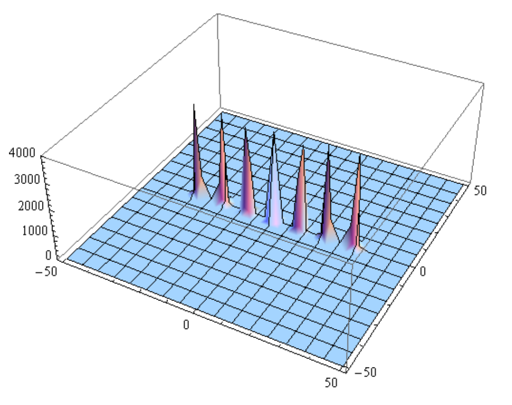
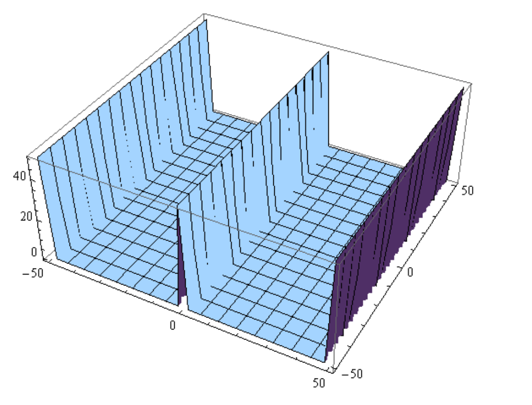
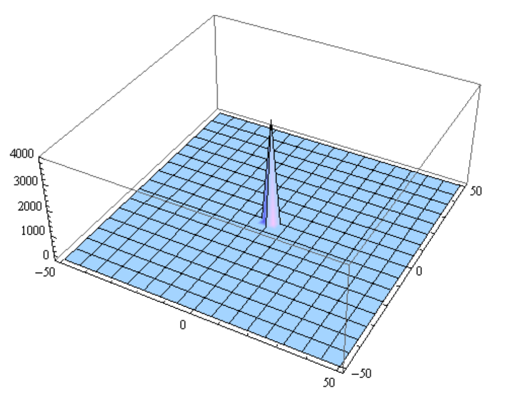
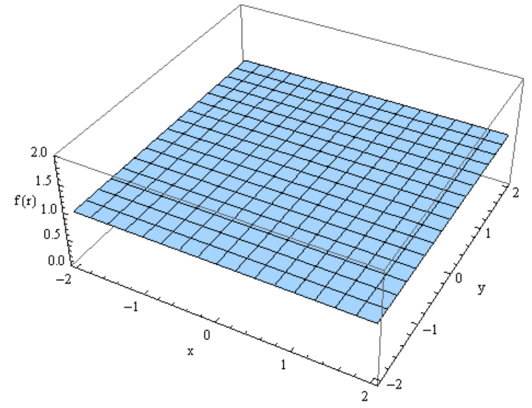
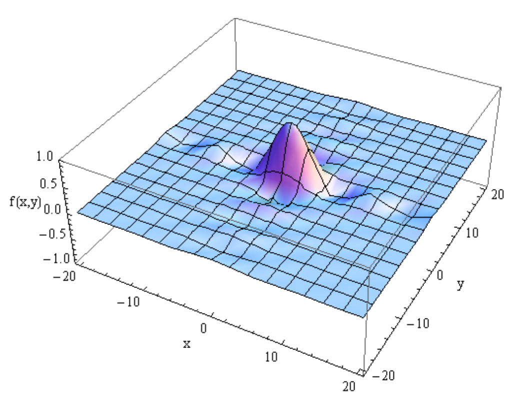
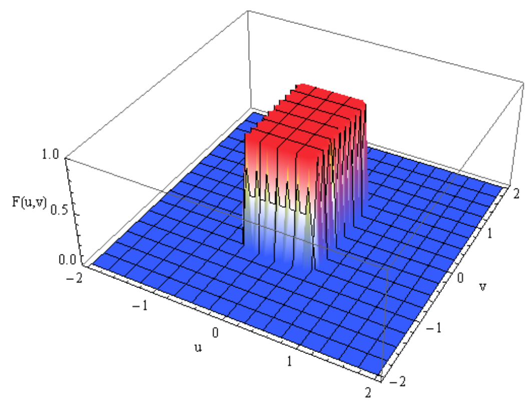
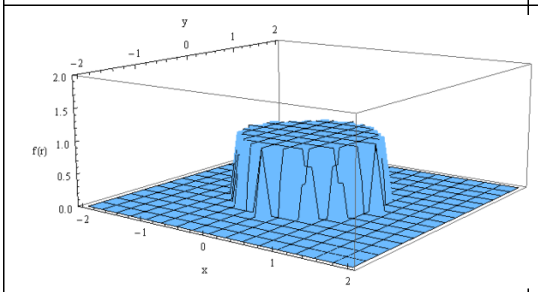
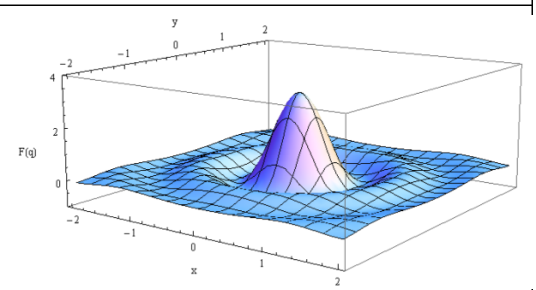
$$\Delta_x f(x,y) = f(x+\frac{1}{2},y) - f(x-\frac{1}{2},y)$$

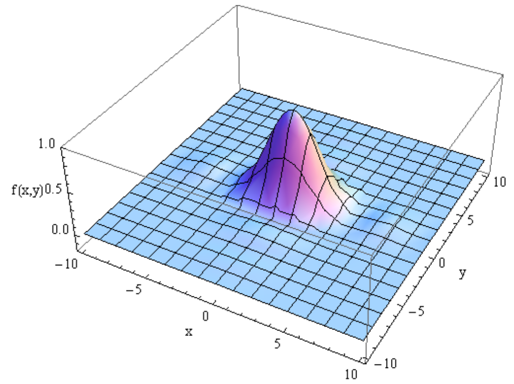
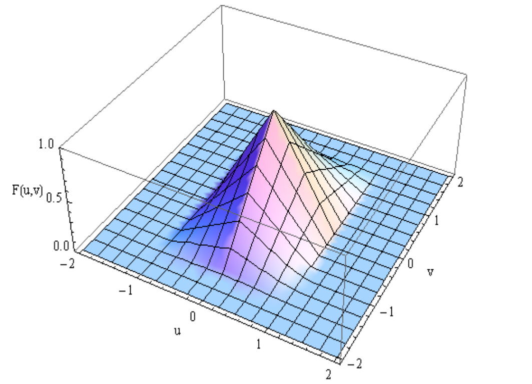
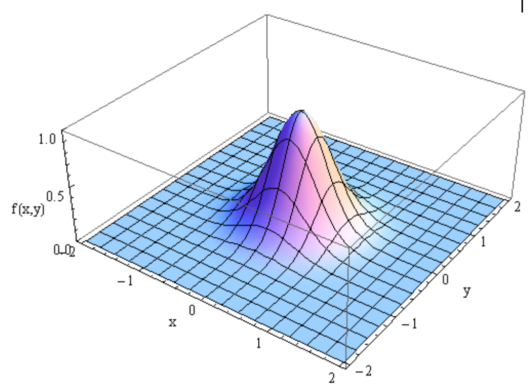
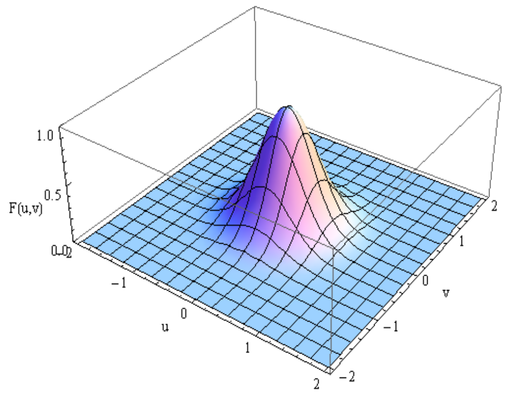
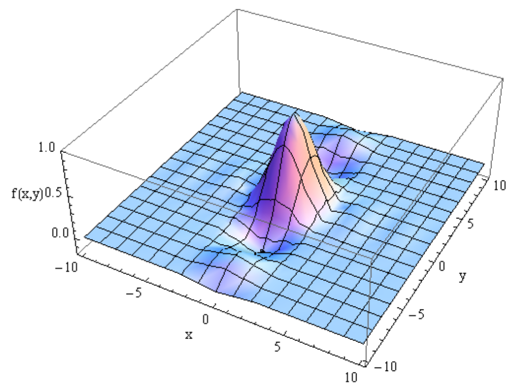
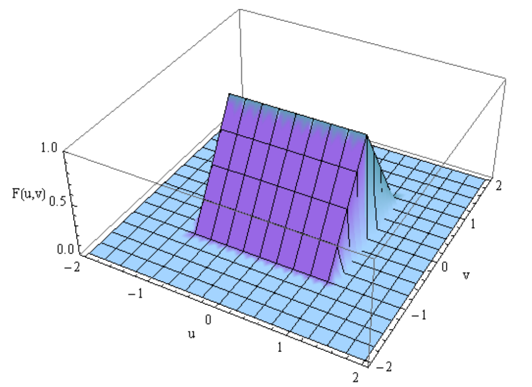
$$\Delta_{xy}^2 f(x,y) = f(x+\frac{1}{2},y+\frac{1}{2}) - f(x-\frac{1}{2},y+\frac{1}{2}) - f(x+\frac{1}{2},y-\frac{1}{2}) + f(x-\frac{1}{2},y-\frac{1}{2})$$

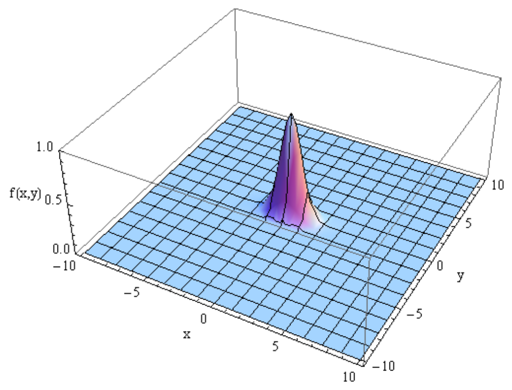
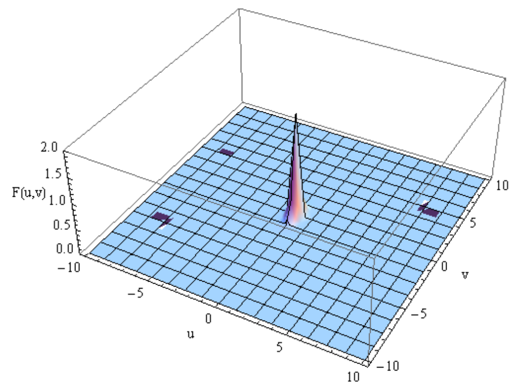
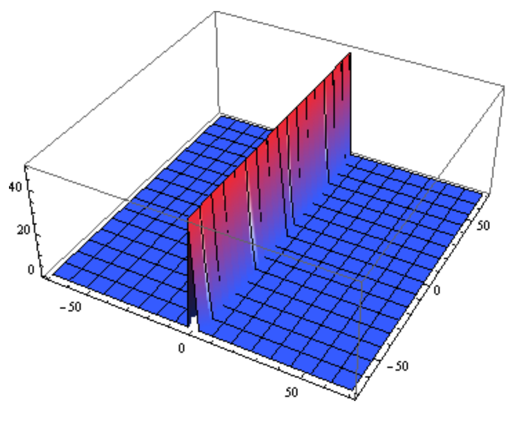
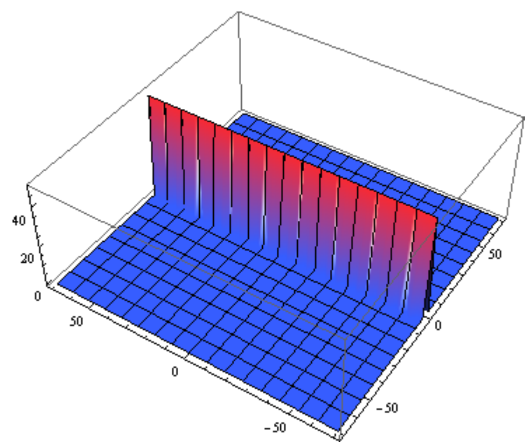
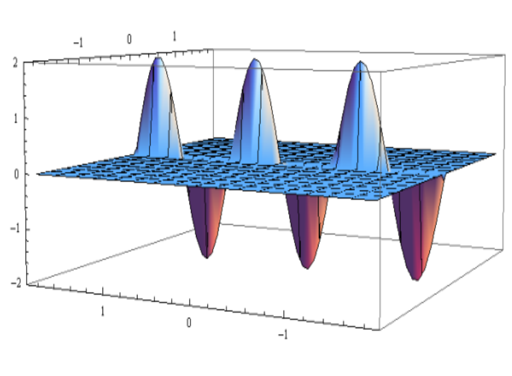
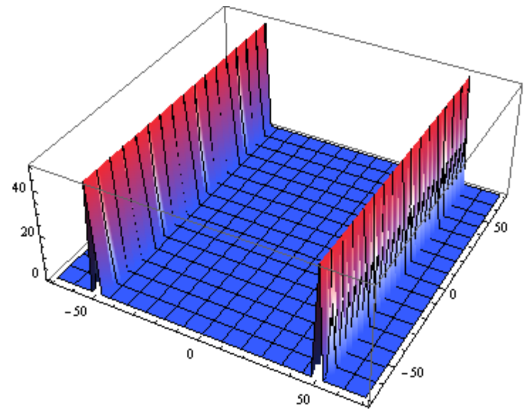
$$\Delta_{xx}^2 f(x,y) = f(x+1,y) - 2f(x,y) + f(x-1,y)$$

The following table illustrates a number of two-dimensional Fourier transforms.

	$f(x,y)$	$F(u,v)$
1	$\cos \pi x$	$\frac{1}{2}\,{}^{2}\delta\left(u-\frac{1}{2},v\right)+\frac{1}{2}\,{}^{2}\delta\left(u-\frac{1}{2},v\right)$
		
2	$\cos 2\pi x \cos 2\pi y$	$\frac{1}{2}\pi\delta(u-2\pi)\delta(v-2\pi)+\frac{1}{2}\pi\delta(u+2\pi)\delta(v-2\pi)$ $+\frac{1}{2}\pi\delta(u-2\pi)\delta(v+2\pi)+\frac{1}{2}\pi\delta(u+2\pi)\delta(v+2\pi)$
		
3	$\mathrm{III}(y)\delta(x)$	$\mathrm{III}(v)$

		
4	$^{2}\delta\left(x,y\right)$	1
		
5	$\operatorname{sinc} x \operatorname{sinc} y$	$^{2}\prod\left(u,v\right)$
		
6	$\prod\left(\frac{\sqrt{x^2+y^2}}{2a}\right)$	$\frac{aJ_1\left(2\pi a\sqrt{u^2+v^2}\right)}{\sqrt{u^2+v^2}}$
		

7	$\text{sinc}^2 x \,\text{sinc}^2 y$	$\Lambda(u)\Lambda(v)$
		
8	$\exp\left[-\pi\left(x^2+y^2\right)\right]$	$\exp\left[-\pi\left(u^2+v^2\right)\right]$
		
9	$\text{sinc}^2 x \,\text{sinc}\, y$	$\Lambda(u)\prod(v)$
		
10	$\exp[-\pi\left(\frac{x^2}{A^2}+\frac{y^2}{a^2}\right)]$	$Aa\exp[-\pi\left(A^2u^2+a^2v^2\right)]$

		
11	$\delta(y)$	$\delta(u)$
		
12	$\cos \pi y \delta(x)$	$\sqrt{\frac{\pi}{2}}\delta(\pi - v) + \sqrt{\frac{\pi}{2}}\delta(\pi + v)$
		

The Hankel Transform of Zero Order [2]

Two-dimensional systems may often show circular symmetry; for example, optical systems are often constructed from components that, in themselves, are circularly symmetrical. Then again, waves spreading out in two dimensions from a source of energy exhibit symmetry for natural reasons. It may be expected that in these cases a simplification will result, for one radial variable will suffice in place of the two independent variables x and y. The appropriate expression of such problems is in terms of the Hankel transform, a one-dimensional transform with Bessel function kernel.

When circular symmetry exists, that is, when

$$f(x,y) = f(r),$$

Where $r^2 = x^2 + y^2$,

Then $F(u,v)$ proves also to be circularly symmetrical; that is.

$$F(u,v) = F(q),$$

Where $q^2 = u^2 + v^2$,

To show this, change the transform formula to polar coordinates and integrate over the angular variable. Then the relations between the two one-dimensional functions $f(r)$ and $F(q)$ are

$$F(q) = 2\pi \int_0^\infty f(r) J_0\left(2\pi qr\right) r dr$$

$$f(r) = 2\pi \int_0^\infty F(q) J_0\left(2\pi qr\right) q dq$$

We refer to $F(q)$ as the Hankel transform(of zero order) of $f(r)$ and note that the transformation is strictly reciprocal, as was the case when the kernels were $\cos$ and $\sin$. The kernel J_0, together with $\cos$, $\sin$, and others, is referred to as a Fourier kernel in the broad sense of a kernel associated with a reciprocal transform.

The factors 2π in the above formulas may be canceled by suitable redefinition of the variables, but their retention follows logically from the form adopted for Fourier transforms. In physical situations the 2π in parentheses will be found to result from the measurement of q in whole cycles per unit of r. The 2π before the integral sign comes from the element of area $2\pi r dr$.

Theorems for the Hankel transform

Theorem	$f(r)$	$F(q)$
Similarity	$f(ar)$	$a^{-2} F\left(\frac{q}{a}\right)$
Addition	$f(r) + g(r)$	$F(q) + G(q)$
Shift	Shift of origin destroys circular symmetry	
Convolution	$\int_0^\infty \int_0^{2\pi} f(r') g(R) r' dr' d\theta$ $(R^2 = r^2 + r'^2 - 2rr'\cos\theta)$	$F(q)G(q)$
Rayleigh	$\int_0^\infty \lvert f(r)\rvert^2 r dr = \int_0^\infty \lvert F(q)\rvert^2 q dq$	

Power	$\int_0^\infty f(r)g^*(r)rdr = \int_0^\infty F(q)G^*(q)qdq$	
Differentiation	$\left(\frac{\partial}{\partial r}\right)^m f(r)$	$(2\pi iq)^m F(q)$
	$\frac{\partial}{\partial r} f(r)$	$2\pi iqF(q)$
	$\frac{\partial^2}{\partial r^2} f(r)$	$-4\pi^2 q^2 F(q)$
Definite integral	$2\pi\int_0^\infty f(r)rdr = F(0)$	
Second moment	$2\pi\int_0^\infty r^2 f(r)rdr = \frac{F''(0)}{-2\pi^2}$	
Equivalent width	$\frac{2\pi\int_0^\infty f(r)rdr}{f(0)} = \frac{F(0)}{2\pi\int_0^\infty F(q)qdq}$	

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