The Swing Voter's Curse: Comment

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In an important paper, Timothy J. Feddersen and Wolfgang Pesendorfer (1996) investigate the "swing voter's curse." In a model of elections in which voters have common preferences and private information, they show that, when indifferent, less informed voters are better off abstaining than voting for either alternative, even without a cost of voting—the "swing voter's curse." They go on to show that even though significant abstention occurs in large electorates, the outcome of the election is almost always the same as with perfect information.

Unfortunately, the proof of the proposition that establishes the "swing voter's curse" phenomenon contains an error. In this comment we identify the error and give a correct proof of the proposition.

I. The Model and Notation

In this section, we sketch the model presented in Feddersen and Pesendorfer (1996) and introduce the appropriate notation. We will follow the notation in Feddersen and Pesendorfer (1996) closely. Interested readers should consult the original paper for more details.

There are $N+1$ potential voters who must, by plurality rule, collectively make a binary choice from the set $\{0,1\}$. We assume N is even, and write $N = 2m$. There are two states of the world, also denoted $\{0,1\}$, and all potential voters share a common prior α that the true state of the world is 0. Each voter is randomly assigned a type by Nature as follows. With probability p_0, the voter prefers that 0 is chosen, regardless of the state of the world. Likewise, with probability p_1, she prefers 1. With probability probability p_i, the voter prefers to select the alternative that matches the true state of the world and with probability q she learns the true state of the world and with probability $1-q$ she does not. Finally, with the remaining probability $p_\phi = 1 - p_0 - p_1 - p_i$, the voter prefers to abstain.

Each of these types have a dominant strategy, except for the uninformed independents [selected with probability $(1-q)p_i$]. Given a symmetric strategy τ for these voters and the dominant strategies of the others, let $\sigma_{z,x}(\tau)$ be the probability that a randomly selected voter votes for alternative x in state z. Then

$$\sigma_{z,x}(\tau) = \begin{cases} p_x + p_i(1-q)\tau_x & \text{if } z \neq x \\ p_x + p_i(1-q)\tau_x + p_i q & \text{if } z = x. \end{cases}$$

From these probabilities can be generated the probability that the other N voters have cast votes that create a tie, $\pi_t(z,\tau)$, and the probability that alternative x receives exactly one less vote than alternative y, denoted $\pi_x(z,\tau)$.[1]

In what follows, we use the following notation:

$$L_j = \frac{(2m)!}{j!j!(2m-2j)!}\sigma_\phi(\tau)^{2m-2j},$$
$$M_j = \frac{(2m)!}{(j+1)!j!(2m-2j-1)!}\sigma_\phi(\tau)^{2m-2j-1},$$

[1]These expressions are given by equations (4) and (5) in Feddersen and Pesendorfer (1996)

$$\Phi = \sum_{j=0}^{m} L_j \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j ,$$

$$\Psi = \sum_{j=0}^{m} L_j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j$$

$$\phi = \sum_{j=0}^{m-1} M_j \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j ,$$

$$\psi = \sum_{j=0}^{m-1} M_j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j .$$

II. Results

In this section we identify the error in Proposition 1 of Feddersen and Pesendorfer (1996) and present a correct proof. We first state the proposition:

Proposition 1 *Let $p_\phi > 0$, $q > 0$, $N \geq 2$ and N even. For any symmetric strategy profile τ in which no agent plays a strictly dominated strategy, $\mathrm{E}\, u(1,\tau) = \mathrm{E}\, u(0,\tau)$ implies $\mathrm{E}\, u(1,\tau) < \mathrm{E}\, u(\phi,\tau)$.*

In other words, if a voter is indifferent between the two alternatives, she strictly prefers to abstain. This is the "swing voter's curse." An implication of this proposition is that there is no Nash equilibrium in which some uninformed independent mixes between voting for 0 and voting for 1.

The second-to-last sentence of the published proof of this proposition states incorrectly that "$\sigma_{1,1}(\tau) = p_i q + \sigma_{1,0}(\tau)$" (p. 419). The correct relationship is that $\sigma_{1,1}(\tau) = p_i q + \sigma_{0,1}(\tau)$. This error makes the line of proof in the paper unsuccessful.[2]

In order to present a correct proof, it will be useful to have the following technical lemma:

Lemma 1

(a). If $\sigma_{1,0}(\tau) = \sigma_{0,1}(\tau)$, *then* $\Phi = \Psi$ *and* $\phi = \psi$.

(b). If $m > 1$ *and* $\sigma_{1,0}(\tau) < (>)\, \sigma_{0,1}(\tau)$, *then* $\phi < (>)\, \psi$. *If* $m = 1$, *then* $\phi = \psi$.

(c). If $\sigma_{1,0}(\tau) < (>)\, \sigma_{0,1}(\tau)$, *then* $\Psi\phi > (<)\, \Phi\psi$.

Proof: As $\sigma_{0,0}(\tau) = \sigma_{1,0}(\tau) + p_i q$ and $\sigma_{1,1}(\tau) = \sigma_{0,1}(\tau) + p_i q$,

$$\begin{aligned}
\Phi &= \sum_{j=0}^{m} L_j \left[\sigma_{1,0}(\tau)\sigma_{0,1}(\tau) + \sigma_{1,0}(\tau)p_i q\right]^j , \\
\Psi &= \sum_{j=0}^{m} L_j \left[\sigma_{1,0}(\tau)\sigma_{0,1}(\tau) + \sigma_{0,1}(\tau)p_i q\right]^j , \\
\phi &= \sum_{j=0}^{m-1} M_j \left[\sigma_{1,0}(\tau)\sigma_{0,1}(\tau) + \sigma_{1,0}(\tau)p_i q\right]^j , \\
\psi &= \sum_{j=0}^{m-1} M_j \left[\sigma_{1,0}(\tau)\sigma_{0,1}(\tau) + \sigma_{0,1}(\tau)p_i q\right]^j .
\end{aligned}$$

Parts (a) and (b) follow immediately from these expressions. To establish

[2] The error does not affect the proofs of any of the other results and, in fact, Proposition 2 (pp. 414–15) implies that for sufficiently large electorates, Proposition 1 holds.

part (c), note that

$$\Psi\phi - \Phi\psi = \left(L_m \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^m + \sum_{j=0}^{m-1} L_j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j \right) \phi$$

$$- \left(L_m \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^m + \sum_{j=0}^{m-1} L_j \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j \right) \psi$$

$$= \left(\sum_{j=0}^{m-1} L_j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j \right) \phi - \left(\sum_{j=0}^{m-1} L_j \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j \right) \psi$$

$$+ \left(L_m \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^m \right) \phi - \left(L_m \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^m \right) \psi$$

$$= \left(\sum_{j=0}^{m-1} L_j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j \right) \left(\sum_{k=0}^{m-1} M_k \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^k \right)$$

$$- \left(\sum_{j=0}^{m-1} L_j \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j \right) \left(\sum_{k=0}^{m-1} M_k \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^k \right)$$

$$+ \left(L_m \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^m \right) \phi - \left(L_m \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^m \right) \psi$$

$$= \left(\sum_{j=0}^{m-1} \sum_{k=0}^{m-1} L_j M_k \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^k \right.$$

$$\left. - \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} L_j M_k \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^k \right)$$

$$+ \left(L_m \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^m \right) \phi - \left(L_m \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^m \right) \psi.$$

We consider the terms in the large parentheses first:

$$\sum_{j=0}^{m-1} \sum_{k=0}^{m-1} L_j M_k \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^k$$

$$- \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} L_j M_k \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^k$$

$$= \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} L_j M_k \left[[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^j [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^k - [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^j [\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^k \right].$$

As the terms with $j = k$ equal zero,

$$= \sum_{\substack{j=0 \\ j \neq k}}^{m-1} \sum_{k=0}^{m-1} L_j M_k \left[[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^j [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^k - [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^j [\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^k \right]$$

$$= \sum_{\substack{j=0 \\ j > k}}^{m-1} \sum_{k=0}^{m-1} L_j M_k \left[[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^j [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^k - [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^j [\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^k \right]$$

$$+ \sum_{\substack{j=0 \\ j < k}}^{m-1} \sum_{k=0}^{m-1} L_j M_k \left[[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^j [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^k - [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^j [\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^k \right]$$

$$= \sum_{\substack{j=0 \\ j > k}}^{m-1} \sum_{k=0}^{m-1} (L_j M_k - L_k M_j) \left[[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^j [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^k \right.$$

$$\left. - [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^j [\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^k \right]$$

$$= \sum_{\substack{j=0 \\ j > k}}^{m-1} \sum_{k=0}^{m-1} \frac{(2m)!(2m)!\sigma_\phi(\tau)^{4m-2k-2j-1}}{(j+1)!j!(2m-2j)!(k+1)!k!(2m-2k)!} [(j+1)(2m-2k) - (k+1)(2m-2j)]$$

$$\times \left[[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^j [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^k - [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^j [\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^k \right]$$

$$= \sum_{\substack{j=0 \\ j > k}}^{m-1} \sum_{k=0}^{m-1} \frac{(2m)!(2m)!\sigma_\phi(\tau)^{4m-2k-2j-1}}{(j+1)!j!(2m-2j)!(k+1)!k!(2m-2k)!} (2m+2)(j-k)$$

$$\times [\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^j [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^k \left[1 - [\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)]^{j-k} [\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)]^{k-j} \right]$$

$$= \sum_{\substack{j=0 \\ j > k}}^{m-1} \sum_{k=0}^{m-1} \frac{(2m)!(2m)!\sigma_\phi(\tau)^{4m-2k-2j-1}}{(j+1)!j!(2m-2j)!(k+1)!k!(2m-2k)!} (2m+2)(j-k)$$

$$\times \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^k \left[1 - \frac{\left[\sigma_{1,0}(\tau)\sigma_{0,1}(\tau) + \sigma_{1,0}(\tau)p_i q\right]^{j-k}}{\left[\sigma_{1,0}(\tau)\sigma_{0,1}(\tau) + \sigma_{0,1}(\tau)p_i q\right]^{j-k}}\right].$$

The last term in brackets is always positive if $\sigma_{1,0}(\tau) < \sigma_{0,1}(\tau)$ and always negative if $\sigma_{1,0}(\tau) > \sigma_{0,1}(\tau)$, and all of the other terms are positive. Thus, the sum is positive if $\sigma_{1,0}(\tau) < \sigma_{0,1}(\tau)$ and negative if $\sigma_{1,0}(\tau) > \sigma_{0,1}(\tau)$.

Returning to the sign of the last two terms of the expression for $\Psi\phi - \Phi\psi$,

$$\begin{aligned}
&\left(L_m \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^m\right)\phi - \left(L_m \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^m\right)\psi \\
&\quad = L_m \left[\left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^m \sum_{j=0}^{m-1} M_j \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j - \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^m \sum_{j=0}^{m-1} M_j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j\right] \\
&\quad = L_m \sum_{j=0}^{m-1} M_j \left[\left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^m \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j - \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^m \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^j\right] \\
&\quad = L_m \sum_{j=0}^{m-1} M_j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^m \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j \left[1 - \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^{m-j} \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^{j-m}\right] \\
&\quad = L_m \sum_{j=0}^{m-1} M_j \left[\sigma_{0,0}(\tau)\sigma_{0,1}(\tau)\right]^m \left[\sigma_{1,0}(\tau)\sigma_{1,1}(\tau)\right]^j \left[1 - \frac{\left[\sigma_{1,0}(\tau)\sigma_{0,1}(\tau) + \sigma_{1,0}(\tau)p_i q\right]^{m-j}}{\left[\sigma_{1,0}(\tau)\sigma_{0,1}(\tau) + \sigma_{0,1}(\tau)p_i q\right]^{m-j}}\right].
\end{aligned}$$

Once again, the last term in brackets is always positive if $\sigma_{1,0}(\tau) < \sigma_{0,1}(\tau)$ and always negative if $\sigma_{1,0}(\tau) > \sigma_{0,1}(\tau)$, and all of the other terms are positive. Thus, the sum is positive if $\sigma_{1,0}(\tau) < \sigma_{0,1}(\tau)$ and negative if $\sigma_{1,0}(\tau) > \sigma_{0,1}(\tau)$. This establishes part (c) of the lemma. ∎

We are now ready to prove the proposition.

Proof of Proposition 1: From equation (8) in Feddersen and Pesendorfer (1996),

$$
\begin{aligned}
\mathrm{E}\,u(1,\tau) - \mathrm{E}\,u(0,\tau) = (1-\alpha)&\left[\pi_t\left(1,\tau\right) + \frac{1}{2}\left(\pi_1\left(1,\tau\right) + \pi_0\left(1,\tau\right)\right)\right] \\
&- \alpha\left[\pi_t\left(0,\tau\right) + \frac{1}{2}\left(\pi_1\left(0,\tau\right) + \pi_0\left(0,\tau\right)\right)\right],
\end{aligned}
$$

and from equations (4) and (5) in Feddersen and Pesendorfer (1996),

$$
\begin{aligned}
&\mathrm{E}\,u(1,\tau) - \mathrm{E}\,u(0,\tau) = \\
&\quad (1-\alpha)\left[\Phi + \frac{1}{2}(\sigma_{1,0}(\tau) + \sigma_{1,1}(\tau))\phi\right] - \alpha\left[\Psi + \frac{1}{2}(\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau))\psi\right].
\end{aligned}
$$

Using the fact that $\sigma_{1,0}(\tau) + \sigma_{1,1}(\tau) = \sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)$,

$$
\begin{aligned}
\mathrm{E}\,u(1,\tau) - \mathrm{E}\,u(0,\tau) = \Phi &+ \frac{1}{2}\left[\sigma_{1,0}(\tau) + \sigma_{1,1}(\tau)\right]\phi \\
&- \alpha\left[\Phi + \Psi + \frac{1}{2}\left[\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)\right](\phi + \psi)\right].
\end{aligned}
$$

Therefore, $\mathrm{E}\,u(1,\tau) - \mathrm{E}\,u(0,\tau) = 0$ implies that

$$
\alpha = \frac{\Phi + \frac{1}{2}\left[\sigma_{1,0}(\tau) + \sigma_{1,1}(\tau)\right]\phi}{\Phi + \Psi + \frac{1}{2}\left[\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)\right](\phi + \psi)}. \tag{1}
$$

From equation (6) in Feddersen and Pesendorfer (1996),

$$
\begin{aligned}
\mathrm{E}\,u(1,\tau) - \mathrm{E}\,u(\phi,\tau) &= \frac{1}{2}\left[(1-\alpha)\left[\pi_t\left(1,\tau\right) + \pi_1\left(1,\tau\right)\right] - \alpha\left[\pi_t(0,\tau) + \pi_1(0,\tau)\right]\right] \\
&= \frac{1}{2}\left[\pi_t\left(1,\tau\right) + \pi_1\left(1,\tau\right) - \alpha\left[\pi_t\left(1,\tau\right) + \pi_t(0,\tau) + \pi_1\left(1,\tau\right) + \pi_1(0,\tau)\right]\right] \\
&= \frac{1}{2}\left[\Phi + \sigma_{1,0}(\tau)\phi - \alpha\left[\Phi + \Psi + \sigma_{1,0}(\tau)\phi + \sigma_{0,0}(\tau)\psi\right]\right]
\end{aligned}
$$

$$= \frac{1}{2}\left[\Phi + \sigma_{1,0}(\tau)\phi - \alpha\left[\Phi + \Psi + \frac{1}{2}(\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau))(\phi + \psi)\right]\right.$$
$$\left. +\alpha\left[\frac{1}{2}(\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau))(\phi + \psi) - (\sigma_{1,0}(\tau)\phi + \sigma_{0,0}(\tau)\psi)\right]\right].$$

Using equation (1) and the fact that $\sigma_{0,0}(\tau) = \sigma_{1,0}(\tau) + p_i q$,

$$= \frac{1}{2}\left[\Phi + \sigma_{1,0}(\tau)\phi - \left[\Phi + \frac{1}{2}(\sigma_{1,0}(\tau) + \sigma_{1,1}(\tau))\phi\right]\right.$$
$$\left. +\alpha\left[\frac{1}{2}(\sigma_{1,0}(\tau) + p_i q + \sigma_{0,1}(\tau))(\phi + \psi) - (\sigma_{1,0}(\tau)\phi + \sigma_{1,0}(\tau)\psi + p_i q\psi)\right]\right]$$
$$= \frac{1}{4}\left[\sigma_{1,0}(\tau)\phi - \sigma_{1,1}(\tau)\phi + \alpha\left[(p_i q + \sigma_{0,1}(\tau) - \sigma_{1,0}(\tau))(\phi + \psi) - 2p_i q\psi\right]\right].$$

Finally, using the fact that $\sigma_{1,1}(\tau) = \sigma_{0,1}(\tau) + p_i q$,

$$= \frac{1}{4}\left[(\sigma_{1,0}(\tau) - \sigma_{0,1}(\tau) - p_i q)\phi + \alpha\left[(\sigma_{0,1}(\tau) - \sigma_{1,0}(\tau))(\phi + \psi) + p_i q(\phi - \psi)\right]\right]$$
$$= \frac{1}{4}\left[(\sigma_{1,0}(\tau) - \sigma_{0,1}(\tau))(\phi - \alpha(\phi + \psi)) - p_i q(\phi - \alpha(\phi - \psi))\right]$$
$$= \frac{1}{4}\left[(\sigma_{1,0}(\tau) - \sigma_{0,1}(\tau))(\phi - \alpha(\phi + \psi)) - p_i q((1 - \alpha)\phi + \alpha\psi)\right]. \quad (2)$$

If $\sigma_{1,0}(\tau) = \sigma_{0,1}(\tau)$, then this equation and Lemma 1 (a) implies $\mathrm{E}\,u(1,\tau) - \mathrm{E}\,u(\phi,\tau) = \frac{1}{4}(-p_i q)\phi < 0$. If $\sigma_{1,0}(\tau) \neq \sigma_{0,1}(\tau)$, then this equation implies $\mathrm{E}\,u(1,\tau) - \mathrm{E}\,u(\phi,\tau) < 0$ if $(\sigma_{1,0}(\tau) - \sigma_{0,1}(\tau))(\phi - \alpha(\phi + \psi)) < 0$. From equation 1,

$$\phi - \alpha(\phi + \psi) = \phi - \frac{\Phi + \frac{1}{2}\left[\sigma_{1,0}(\tau) + \sigma_{1,1}(\tau)\right]\phi}{\Phi + \Psi + \frac{1}{2}\left[\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)\right](\phi + \psi)}(\phi + \psi).$$

As $\sigma_{1,0}(\tau) + \sigma_{1,1}(\tau) = \sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)$, we obtain equation (3), below.

$$
\begin{aligned}
&= \frac{\phi\left[\Phi + \Psi + \frac{1}{2}\left[\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)\right](\phi + \psi)\right] - (\phi + \psi)\left[\Phi + \frac{1}{2}\left[\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)\right]\phi\right]}{\Phi + \Psi + \frac{1}{2}\left[\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)\right](\phi + \psi)} \\
&= \frac{\left[\Phi + \Psi\right]\phi - \Phi\left[\phi + \psi\right]}{\Phi + \Psi + \frac{1}{2}\left[\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)\right](\phi + \psi)} \\
&= \frac{\Psi\phi - \Phi\psi}{\Phi + \Psi + \frac{1}{2}\left[\sigma_{0,0}(\tau) + \sigma_{0,1}(\tau)\right](\phi + \psi)}. \qquad (3)
\end{aligned}
$$

The denominator of this expression is positive, and thus by Lemma 1 (c), $(\sigma_{1,0}(\tau) - \sigma_{0,1}(\tau))(\phi - \alpha(\phi + \psi))$ is negative. Thus, in each case, $\mathrm{E}\, u(1, \tau) - \mathrm{E}\, u(\phi, \tau) < 0$ and the proposition is established. ∎

References

Feddersen, Timothy J. and Wolfgang Pesendorfer, "The Swing Voter's Curse," *American Economic Review*, June 1996, *86* (3), 408–424.