A Note on the Condorcet Jury Theorem with Supermajority Voting Rules

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1 Introduction

The Condorcet Jury Theorem has been the subject of a extensive literature over the past decades. In its original formulation, the Condorcet Jury Theorem states that a majority of a group is more likely than a single individual to choose the better of two alternatives (Black, 1958; McLean and Hewitt, 1994). Recent work on jury theorems has generalized this result in a number of ways. Ladha (1992) considers the case of correlated information. Miller (1986) considers a jury theorem in the context of an electorate divided into two political parties. Boland (1989) provides a version of the theorem for heterogeneous groups and indirect systems. In a series of papers, Paroush and his co-authors establish sufficient conditions (Paroush, 1998), necessary and sufficient conditions (Berend and Paroush, 1998), and a nonasymptotic jury theorem (Ben-Yashar and Paroush, 2000).

Jury theorems with supermajority rules have been considered by only a few papers in the literature. Nitzan and Paroush (1984) and Ben-Yashar and Nitzan (1997) establish that supermajority rules (which they term "qualified majority rules") maximize the probability of making a correct decision among the class of decisive rules, where a decisive rule is one which always selects an alternative. For supermajority rules, this means that one of the two choices (the "status quo") is privileged in that it is chosen unless it is overturned by the given supermajority. This differs from our setting, in which no decision is made if neither choice receives the required supermajority and both making the wrong decision and making no decision are equally bad. This setting is considered by Kanazawa (1998), but as mentioned below, the result presented for heterogenous groups is incorrect.

It is important to note that this literature on the "classic" Condorcet Jury Theorem assumes voters act sincerely or "informatively" in their voting decision and ignores the inference that a fully rational voter should make based on being pivotal (Austen-Smith and Banks, 1996; Feddersen and Pesendorfer, 1996). In this context, jury theorems for supermajority rules, including unanimity, have been studied by several authors (Feddersen and Pesendorfer, 1998; Duggan and Martinelli, 2001).

In this note, we examine the question of jury theorems with supermajority rules. We give two theorems, one dealing with homogeneous groups (in which voters are identically competent) and the other dealing with heterogeneous groups (in which voters differ in their ability to perceive which alternative is better). We show that, in large enough electorates, a jury theorem holds as long as the average competence of the voters is greater than the fraction of votes needed for passage.

2 The Model

Let n be the number of voters and let the probability that voter i votes correctly be p_i, with $0 \leq p_i \leq 1$. We denote the vector of these probabilities as $p = (p_1, \ldots, p_n)$. Formally, for $i = 1, \ldots, n$, let X_i be an independent Bernoulli random variable with parameter p_i, and let $X = X_1 + \cdots + X_n$. In addition, for a given vector p, let $\bar{p} = \sum_1^n p_i/n$, the average probability of a correct decision. Let Y be a binomial random variable with parameters n and $\bar{p}$. We interpret the collection of random variables $\{X_i\}$ as a heterogeneous group with competence vector p. Similarly, Y can be viewed as a homogeneous group with the same average competence.

Let q be the fraction of the electorate required to choose an alternative. That is, at least $\lceil qn \rceil$ voters are necessary to make a collective decision, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Such a supermajority rule is also known as a q-rule. Unanimity rule corresponds to $q = 1$ and majority rule corresponds to $q = 1/2 + 1/n$. Thus, the probability that a group of size n makes the correct decision using a q-rule can be expressed formally as

$$P_{q,n} = P[X \geq qn].$$

Similarly, we define the probability that a homogeneous group of equal average competence makes the correct decision as

$$R_{q,n} = P[Y \geq qn],$$

where Y is the distribution defined above.[1]

3 Results

We first present the original formulation of the Condorcet Jury Theorem, namely majority rule in homogeneous groups. For a proof, see Miller (1986). Also, Boland (1989) shows a similar result holds for the case of heterogeneous groups under majority rule.

Theorem 1 (Miller) *Assume that* $p_1 = \cdots = p_n = p$ *and* $q = 1/2 + n/2$. *Denote* $P_{1/2+n/2,n}$ *by* P_n.

1. *If* $1/2 < p < 1$ *and* $n \geq 3$, *then* $P_n > p$ *and* $P_n \to 1$ *as* $n \to \infty$.
2. *If* $0 < p < 1/2$ *and* $n \geq 3$, *then* $P_n < p$ *and* $P_n \to 0$ *as* $n \to \infty$.
3. *If* $p = 1/2$, *then* $P_n = 1/2$ *for all* n.

In the case of supermajority rules, we show that similar results hold for sufficiently large n. We first state our result for homogeneous groups.

Theorem 2 *Assume that* $p_1 = \cdots = p_n = p$ *and* $q > 1/2$.

1. *If* $p > q$, *then there exists an integer* N *such that for all* $n > N$, $P_{q,n} > p$ *and* $\lim_{n\to\infty} P_{q,n} = 1$.

[1] A correct decision is a decision in which the correct alternative is actually chosen. Situations in which neither alternative receives enough votes to be chosen, as well as in which the wrong alternative is chosen, are not considered correct decisions.

2. *If* $p < q$*, then there exists an integer* N *such that for all* $n > N$*,* $P_{q,n} < p$ *and* $\lim_{n\to\infty} P_{q,n} = 0$*.*

3. *If* $p = q$*, then there exists an integer* N *such that for all* $n > N$*,* $P_{q,n} < p$ *and* $\lim_{n\to\infty} P_{q,n} = 0$*.*

Proof: For part (1), given $p > q$, $\delta = p - q > 0$. By the Weak Law of Large Numbers, for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $n > N(\varepsilon)$,

$$P[|\, X/n - p\,| > \delta] \;<\; \epsilon.$$

As $P_{q,n} = P[X/n \geq q]$, it follows that

$$P_{q,n} \;\geq\; 1 - P[|\, X/n - p\,| > \delta] \;>\; 1 - \varepsilon.$$

Setting $\varepsilon = 1 - p$, there exists an integer N such that for all $n > N$, $P_{q,n} > p$. Letting $\varepsilon \to 0$, it follows that $\lim_{n\to\infty} P_{q,n} = 1$.

Part (2) follows by the same line of reasoning by setting $\delta = q - p > 0$ and noting that

$$P_{q,n} \;\leq\; P[|\, X/n - p\,| > \delta] \;<\; \varepsilon.$$

For part (3), by the Central Limit Theorem,

$$P[X/n \geq q] \;=\; P[X/n \geq p] \;\to\; 1/2,$$

as $n \to \infty$. As $q > 1/2$, for sufficiently large n, $P_{q,n} < p$. ∎

This result establishes that a jury theorem holds for sufficiently large n, as long as $p > q$. It is easy to show that this theorem does not holds for all n. For example, if $n = 100$, $q = 2/3$ and $p = 0.673$, a calculation of the appropriate binomial distribution yields $P_{2/3,100} = .5752$.

We now consider heterogeneous groups. As in Boland (1989) and Kanazawa (1998), our result draws on a theorem of Hoeffding (1956).[2] We prove that a jury theorem holds (fails) for sufficiently large heterogeneous groups if the average competence is greater (less) than the fraction required for choice.[3]

Theorem 3 *Assume that $q > 1/2$.*

1. *If $\bar{p} > q$, then there exists an integer N such that for all $n > N$, $P_{q,n} > \bar{p}$ and $\lim_{n\to\infty} P_{q,n} = 1$.*

2. *If $\bar{p} < q$, then there exists an integer N such that for all $n > N$, $P_{q,n} < \bar{p}$ and $\lim_{n\to\infty} P_{q,n} = 0$.*

Proof: Hoeffding (1956, Theorem 5) establishes that if b and c are integers such that

$$0 \;\le\; b \;\le\; n\bar{p} \;\le\; c \;\le\; n,$$

then

$$P[b \le Y \le c] \;\le\; P[b \le X \le c],$$

where Y is a binomial random variable with parameters n and $\bar{p}$.

For part (1), as $\bar{p} > q$, $n\bar{p} > \lceil nq \rceil$ holds for all $n > 1/(\bar{p} - q)$. Setting $b = \lceil nq \rceil$ and $c = n$, it is immediate from the above that if $n > 1/(\bar{p} - q)$, $P_{q,n} \ge R_{q,n}$. As part 1 of Theorem 2 applies to $R_{q,n}$, the same results hold for $P_{q,n}$.

For part (2), note that $\bar{p} < q$ implies that $n\bar{p} \le \lceil nq \rceil - 1$ holds for all $n \ge 1/(\lceil q \rceil - \bar{p})$. Setting $b = 0$ and $c = \lceil nq \rceil - 1$, we see that, for sufficiently

[2] An alternative method of proof is to use a weak law of large numbers for nonidentically distributed random variables (Feller, 1968, X.5).

[3] This corrects an erroneous result in Kanazawa (1998) asserting that a jury theorem holds when $\bar{p} \ge \pi(n+1)/n$.

large n,

$$P[0 \le Y \le \lceil nq \rceil - 1] \ \le \ P[0 \le X \le \lceil nq \rceil - 1],$$

which yields

$$P[0 \le Y < \lceil nq \rceil] \ \le \ P[0 \le X < \lceil nq \rceil].$$

This is equivalent to $P_{q,n} \le R_{q,n}$. Combining this with part 2 of Theorem 2 completes the proof. ∎

Attentive readers will note that we do not address the case of $\bar{p} = q$ in this theorem. This is because no general result holds in this case. From part 3 of Theorem 2, we know that, for sufficiently large n, a homogeneous group yields $P_{q,n} < \bar{p}$. On the other hand, with the simplifying assumption that $n\bar{p} = nq$ is an integer, a group of size n with $p_1 = \cdots = p_{nq} = 1$ and $p_{qn+1} = \cdots = p_n = 0$ has $P_{q,n} = 1$. Therefore no general result holds, even for large n.

4 Conclusion

In this paper, we have shown that for sufficiently large electorates, a Condorcet jury theorem holds for supermajority rules. In particular, we show that if the average competence of the voters is greater than the fraction of votes needed for passage, then in sufficiently large groups, a group decision is more likely to be correct than the decision of a single randomly chosen individual.

The institutional design question of what value of q is optimal has been addressed by Nitzan and Paroush (1984) and Ben-Yashar and Nitzan (1997) in the context of decisive rules, with the requisite notion of a status quo alternative. In the present model, it is possible for neither alternative to receive a supermajority, an outcome which we assume is exactly as undesirable as choosing the wrong alternative. If instead we suppose that making the wrong decision is worse than reaching no decision, we are faced with a new tradeoff for choosing an optimal q. The reason is that while the probability of a correct decision is decreasing in q, so is the probability of making an incorrect decision.[4] We leave this to future work.

[4]See Feddersen and Pesendorfer (1998) for an examination of this issue in a model with fully rational voters.

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