Group Support and Top-Heavy Rules

Mark Fey*

Abstract

This paper uses an axiomatic approach to study the properties of voting procedures. We propose a new axiom, called "group support," which requires that an alternative that is selected by a society must be selected by some subgroup of the society. We show that group support and faithfulness characterize a class of scoring rules that we call top-heavy rules.
JEL Classification: D71
Keywords: scoring rules, group support, faithfulness, plurality rule

1 Introduction

One approach to understanding methods of making group choice is by characterizing various choice procedures by the sets of axioms that they uniquely satisfy. Researchers have developed such characterizations for specific rules, such as majority rule (May, 1952), Borda's rule (Young, 1974), and plurality rule (Richelson, 1978; Ching, 1996), as well as broad classes of rules such as scoring rules (Smith, 1973; Young, 1975).

In this spirit, we provide a characterization of a particular group of scoring rules in terms of faithfulness and a new axiom that we call group support. Faithfulness, introduced by Young (1974), requires that the social choice from a single individual be her top-ranked alternative. Group support states that if an alternative is chosen by society, then it must be

*My thanks to John Duggan and Bahar Leventoglu for their comments. Mailing address: Department of Political Science, Harkness Hall, University of Rochester, Rochester, NY, 14627-0146. Phone: (585) 275-5810. Fax: (585) 271-1616. E-mail address: markfey@mail.rochester.edu.

possible to divide the society into two groups, one of which would choose that alternative. The class of scoring rules that we characterize is termed top-heavy rules as they assign heavy weight to top-ranked alternatives and those close to the top of voters' preference orders.

As plurality rule is a prime example of a top-heavy rule, our result is most similar to the characterizations of plurality rule with variable electorate by Richelson (1978) and Ching (1996). However, these papers impose an axiom which states that removing Pareto-dominated alternatives does not alter the social choice. By using faithfulness instead, we weaken this property by requiring that it only hold for single member societies. On the other hand, of course, we add the group support axiom that is not present in these papers.

2 Notation and Definitions

Let $A = \{a_1, \dots a_m\}$ be the set of *alternatives* and let the set of potential *voters* be $\mathbb{N} = \{1, 2, \dots\}$, the set of natural numbers. A *society* of size n is then the set $\{1, 2, \dots, n\}$. Every voter i in $\mathbb{N}$ has *preferences* that we assume are given by a linear order on the alternative set A. Let $\mathcal{P}$ be the set of all linear orders on A. Then voter i's preference order is denoted $P_i \in \mathcal{P}$ and a *profile* of size n is given by the vector $P^n = (P_1, \dots, P_n)$. For later use, we denote the top-ranked alternative of a linear order by $T(P_i)$.

A *social choice function* f is a function that assigns to every profile of every possible size a nonempty subset of A. Let $\mathcal{A}$ denote the set of all nonempty subsets of A. Then f is a function $f : \bigcup_{n=1}^{\infty} P^n \to \mathcal{A}$.

We now specify the axioms we consider.

Faithfulness If $n = 1$, then $f(P_1) = T(P_1)$.

This condition states that if society consists of only a single individual, then the social choice is just the individual's top-ranked alternative.

Group Support For any profile P^n, if $a_j \in f(P^n)$, then there exists a two-element partition of P^n, P' and P'', both nonempty, such that $a_j \in f(P')$ or $a_j \in f(P'')$.

This axiom states that if an alternative is chosen by society, then it must be possible to divide the society into two groups, one of which would choose that alternative. Strengthening this condition, we have the following:

Individual Support For any profile P^n, if $a_j \in f(P^n)$, then there exists an individual i such that $a_j \in f(P_i)$.

Now consider a social choice function that satisfies group support and fix an alternative that is chosen by society. We can find a subgroup that also chooses the given alternative. Now, applying group support to this subgroup, we can find an even smaller subgroup that supports this alternative. By continuing to apply group support to these smaller and smaller groups, we must eventually arrive at a single individual who supports the social choice. From this we conclude that, in fact, a social choice function satisfies group support if and only if it satisfies individual support.

3 Top-Heavy Rules

The rules we are interested in belong to the class of social choice functions known as scoring rules. Formally, we say that a social choice function is a *scoring rule* if there a set of m values $s_1 \geq s_2 \geq \cdots \geq s_m$ such that every voter assigns score s_j to her jth-ranked alternative, and the function selects those alternatives with the maximal total scores. For a profile of size n, let $r_j(a, P^n)$ be the number of voters that have alternative a jth-ranked in P^n. For example, $r_1(a, P^n)$ is the number of voters with a top-ranked. We define $s(a, P^n)$, the *score* of alternative a in profile P^n, by

$$s(a, P^n) = \sum_{j=1}^{m} s_j r_j(a, P^n).$$

Then we say a social choice function f is a scoring rule if it satisfies

$$f(P^n) = \{a \in A \mid s(a, P^n) \geq s(a_k, P^n) \text{ for } k = 1, \ldots, m\},$$

for all profiles P^n.

The *plurality rule* is a social choice function that selects exactly those alternatives that are top-ranked by the largest number of voters. It is easy to see that this rule is an example of a scoring rule in which $s_1 = 1$ and $s_2 = s_3 = \ldots s_m = 0$. In fact, plurality rule can be represented by any scoring rule with scores satisfying $s_1 > s_2 = s_3 = \cdots = s_m$.

More generally, we are interested in a class of rules that we term *top-heavy rules*, which are scoring rules that place heavy weight on alternatives at or near the top of voters' preference orders and relatively little weight on lower-ranked alternatives. Plurality rule is an extreme case of this requirement, as it depends entirely on individuals' top-ranked alternatives and ignores the ranking of alternatives that are not top-ranked. To be precise, we say a scoring rule f is a top-heavy rule if it satisfies

$$\frac{\sum_{j=1}^{m} s_j}{m} > s_2. \tag{1}$$

To understand this expression, observe that $\sum_{j=1}^{m} s_j$ is the sum of scores from a single voter. Thus, for a top-heavy rule, the average score across all alternatives must be greater than the score assigned to a second-ranked alternative. Intuitively, this requires that top-ranked alternatives be assigned a disproportionately large score. Clearly, plurality rule is an example of a top-heavy rule. On the other hand, Borda's rule is an example of a well-known scoring rule that is not top-heavy.

We are now ready to state our main result.

Theorem 1 *A scoring rule f is a top-heavy rule if and only if it satisfies group support and faithfulness.*

Proof: Suppose f is a scoring rule. We first show that if f satisfies group support and faithfulness, then equation 1 holds. Group support is equivalent to individual support and this condition, together with faithfulness, implies that an alternative that is not top-ranked by any voter in a society cannot be chosen. Now consider the following profile of size $m - 1$:

$$\begin{array}{cccc}
\underline{P_1} & \underline{P_2} & \cdots & \underline{P_{m-1}} \\
a_1 & a_2 & \cdots & a_{m-1} \\
a_m & a_m & \cdots & a_m \\
a_2 & a_3 & \cdots & a_1 \\
a_3 & a_4 & \cdots & a_2 \\
\vdots & \vdots & \cdots & \vdots \\
a_{m-1} & a_1 & \cdots & a_{m-2}
\end{array}$$

In this case, a_m cannot be chosen. Clearly, $s(a_m) = (m-1)s_2$ and $s(a_j) = s_1 + s_3 + \cdots + s_m$, for $j = 1, \ldots, m-1$. From this it follows that $\sum_{j=1}^m s_j > ms_2$, which is equation 1.

To prove the reverse direction of the theorem, we begin by showing that a top-heavy rule satisfies faithfulness. A scoring rule is faithful if and only if it satisfies $s_1 > s_2$. As s_1 is a maximum of $\{s_1, \ldots, s_m\}$ and the average of a set of values is never larger than its maximum, it follows that $s_1 \geq \sum_{j=1}^m s_j/m > s_2$ for a top-heavy rule, so it is faithful.

We now show that a top-heavy rule satisfies individual support and thus group support. Suppose not. Then, because a top-heavy rule satisfies faithfulness, there exists a profile P^n in which some alternative a is chosen without being top-ranked by any voter. Therefore, $s(a) \leq ns_2$ and the total points assigned to all alternatives other than a can be no less than $n\sum_{j=1}^m s_j - ns_2$. Of course, some alternative from this group must receive at least the average of the total points assigned, so

$$s(a') \geq \frac{n(\sum_{j=1}^m s_j - s_2)}{m-1},$$

for some alternative $a' \neq a$. As a is chosen, $s(a) \geq s(a')$, so

$$\begin{aligned}
ns_2 &\geq \frac{n(\sum_{j=1}^m s_j - s_2)}{m-1} \\
(m-1)s_2 &\geq \sum_{j=1}^m s_j - s_2.
\end{aligned}$$

This contradicts equation 1, so our proof is complete. ∎

While this theorem characterizes top-heavy rules within the class of scoring rules, it is not difficult to give a characterization of top-heavy rules within the class of *all* social choice functions. To do so, we rely on a powerful result of Young (1975) that characterizes scoring rules by the following five axioms: anonymity, neutrality, reinforcement, overwhelming majority, and monotonicity. Anonymity and neutrality require that individuals and alternatives be treated equally. Under reinforcement, if an alternative is chosen by two societies separately, then it is chosen when the two societies are joined together. Overwhelming majority states that if one society chooses a particular alternative, then any (larger) society which contains a sufficiently large replication of it must also choose that alternative. Finally, monotonicity requires that a chosen alternative that is raised in the preference orders of voters should still be chosen.[1] The following corollary is immediate.

Corollary 1 *A social choice function f is a top-heavy rule if and only if it satisfies anonymity, neutrality, reinforcement, overwhelming majority, monotonicity, group support, and faithfulness.*

[1]This axiom is needed only to insure that the scores satisfy $s_1 \geq \cdots \geq s_m$. In fact, it is possible to drop the monotonicity axiom from this characterization with only minor modifications of the proof of Theorem 1.

References

Ching, Stephen 1996. "A simple characterization of plurality rule," *Journal of Economic Theory*, 71, 298–302.

May, Kenneth O. 1952. "A set of independent necessary and sufficient conditions for simple majority decisions," *Econometrica*, 20, 680–684.

Richelson, Jeffrey T. 1978. "A characterization result for the plurality rule," *J. Econom. Theory*, 19, 548–550.

Smith, John H. 1973. "Aggregation of preferences with variable electorate," *Econometrica*, 41, 1027–1041.

Young, H. P. 1974. "An axiomatization of Borda's rule," *J. Econom. Theory*, 9, 43–52.

Young, H. P. 1975. "Social choice scoring functions," *SIAM J. Appl. Math.*, 28, 824–838.