An Undecidable Statement Regarding Zero-Sum Games

Mark Fey*

Abstract

In this paper, we give an example of a statement concerning two-player zero-sum games which is undecidable, meaning that it can neither be proven or disproven by the standard axioms of mathematics. Earlier work has shown that there exist "paradoxical" two-player zero-sum games with unbounded payoffs, in which a standard calculation of the two players' expected utilities of a mixed strategy profile yield a positive sum. We show that whether or not a modified version of this paradoxical situation, with bounded payoffs and a weaker measurability requirement, exists is an unanswerable question. Our proof relies on a mixture of techniques from set theory and ergodic theory.

*Department of Political Science and Wallis Institute of Political Economy, University of Rochester. email: mark.fey@rochester.edu

1 Introduction

One strength of game theory lies in its mathematical foundation, which provides a rigorous framework for analyzing complex strategic interactions between rational decision-makers. This mathematical foundation has several advantages. It provides a high level of rigor, ensuring that conclusions drawn from the theory are logically sound. It provides clarity, so arguments are easier to understand and evaluate. It allows for independent verification, as others can check the correctness of mathematical arguments. Finally, it can suggest new insights into problems by opening up new avenues of research and inquiry.

However, there are limits to what can be achieved by a mathematical theory. Gödel's celebrated Incompleteness Theorem shows that there exist mathematical statements which can neither be proved nor disproved. A concrete example of such a statement is the Continuum Hypothesis, which states that every infinite set of real numbers is either countable or has the same cardinality as the real line. Famously, the work of Gödel (1940) and Cohen (1963, 1964) shows that the Continuum Hypothesis can neither be proved true or false using the standard axioms of mathematics. In other words, the Continuum Hypothesis is undecidable.

In this paper, we give a statement regarding two-player zero-sum games that is undecidable, in the same manner as the Continuum Hypothesis. The specific statement we consider relates to calculating the expected utility of mixed strategies in two-player zero-sum games. In order to understand the importance of this statement, consider the standard approach to calculate the expected utility of a profile of mixed strategies (σ_1, σ_2). The first step is often to calculate the expected utility of each of a player's pure strategies, given the mixed strategy of the opponent, which is given by $U_i(s_i \mid \sigma_j) = \int u_i(s_i, s_j)\, d\sigma_j(s_j)$. Then, using these expected utilities, the expected utility of the profile of mixed strategies (σ_1, σ_2) is calculated as $\int U_i(s_i \mid \sigma_j)\, d\sigma_i(s_i)$. Alternatively, the functions $U_i(s_i \mid \sigma_j)$ may be used directly in identifying an equilibrium. This is done by checking that each player is indifferent across all of the pure strategies in the support of her own mixed strategy and weakly prefers these pure strategies to those outside the support.

However, as noted by Baye, Kovenock and De Vries (2012), this standard approach can lead to paradoxical results when applied to games with unbounded payoff functions. The authors provide an example of an auction game with unbounded action sets and correspondingly unbounded payoffs in which both players can receive net positive equilibrium payoffs even though only pure transfers are available in the game. More specifically, they show that there exists a measurable zero-sum utility function $u_1 = -u_2$ and a mixed strategy profile (σ_1, σ_2)

such that $U_i(s_i \mid \sigma_j)$ exists and is finite for every $s_i \in S_i$, but

$$\int U_1(s_1 \mid \sigma_2)\, d\sigma_1(s_1) + \int U_2(s_2 \mid \sigma_1)\, d\sigma_2(s_2) > 0.$$

In other words, the sum of the expected payoffs to the two players in a zero-sum game can be positive-sum! In Baye, Kovenock and De Vries (2012), this is dubbed the "Herodotus paradox," as the example comes from the historian Herodotus' account of Babylonian bridal auctions. As discussed in Baye, Kovenock and De Vries (2012), this apparent paradox occurs because an unbounded utility function can fail to be integrable with respect to the joint measure $\sigma_1 \otimes \sigma_2$. In such a case, the standard approach described above is incorrect. The authors also mention that this paradox cannot occur if the player's strategy sets are compact and the zero-sum utility function is measurable and bounded.

Essentially, the statement we consider asks if the Herodotus paradox can occur in a zero-sum game with compact strategy sets and a bounded utility function, but with a weaker measurability requirement. More specifically, we ask, in the context of two-player zero-sum games on the unit square, is there a bounded utility function $u_1 = -u_2$ and a mixed strategy profile (σ_1, σ_2) such that u_i is measurable in s_i, holding fixed s_j, $U_i(s_i \mid \sigma_j)$ is measurable, and

$$\int U_1(s_1 \mid \sigma_2)\, d\sigma_1(s_1) + \int U_2(s_2 \mid \sigma_1)\, d\sigma_2(s_2) \neq 0.$$

The precise statement is given in Section 3 as Statement A.

Our main result is that this statement is undecidable, so that it can neither be proven nor disproven by the standard axioms of mathematics. That is, whether or not there exists a certain kind of paradoxical two-player zero-sum game on the unit square is an unanswerable question. It may seem challenging to prove that something is unprovable, but our approach is fairly straightforward. In particular, we find two undecidable statements, the first of which implies that Statement A is true and the second of which implies that Statement A is false. Specifically, by using a construction that dates back to Sierpiński (1920), we first show that if the Continuum Hypothesis holds, then Statement A is true. Then we show that if a specific undecidable statement about two cardinal characteristics of the continuum holds, then statement A is false. Putting these two implications together implies that statement A is undecidable.

To our knowledge, statement A is the first undecidable statement relating to strategic form games.[1] As in Baye, Kovenock and De Vries (2012), the ultimate source of our results is whether or not we can apply Fubini's Theorem for calculating the expected utility of mixed strategies with respect to the product measure of the mixtures. Thus our work is closely related to results in pure mathematics on undecidable versions of Fubini's Theorem (Friedman, 1980; Freiling, 1986; Shipman, 1990; Recław and Zakrzewski, 1999). We extend this work by showing how it intersects with statements about zero-sum games and by offering a different method of proof using ergodic theory.

We should also mention that other connections between games and axiomatic set theory exist in the mathematical literature. One specific example is Prikry and Sudderth (2016). This paper also deals with two-player zero-sum games. Specifically, the authors show that whether or not the upper value function of a parameterized utility function is universally measurable is undecidable. More broadly, there is a rich literature on topological games (Berge, 1957). These are sequential games in which two players alternate making choices from some mathematical set. These choices are formed into a sequence, with the winner determined by whether or not the sequence has some property. There are many varieties of topological games (Kechris, 1995). While most interesting topological games are *determined*, meaning a winning strategy for one player always exists, it is possible using the Axiom of Choice to construct topological games that are undetermined. As an alternative to the Axiom of Choice, the Axiom of Determinacy states that every set of reals is determined. Variations of the axiom are useful in showing what mathematical results can be proven using only limited versions of the Axiom of Choice. Litak (2018) examines some interesting applications of these axioms to social choice theory and inter-generational utility aggregation.

2 Mathematical Preliminaries

2.1 Sets and Cardinality

We begin with some basic facts from set theory.[2] Consider a nonempty set X. The binary relation $<$ on X is a *linear order* if it is irreflexive, antisymmetric, and transitive.[3] If $<$ is a linear order on X, we say X is *ordered* by $<$. We write $x \leq y$ if $x < y$ or $x = y$. For a subset

[1]A different use of the term "undecidable" refers to computational questions. See, for example, Rabin (1957), Rubinstein (1998, Ch. 10), and Velupillai (2009).

[2]For more details, see Jech (2003) or Ciesielski (1997).

[3]A binary relation R on X is irreflexive if xRx holds for no $x \in X$, antisymmetric if for all $x, y \in X$ with $x \neq y$, either xRy or yRx holds, and transitive if for all $x, y, z \in X$, xRy and yRz imply xRz.

$A \subseteq X$ ordered by $<$, an element a is the *smallest element* of A if $a \in A$ and $a \leq x$ for all $x \in A$. A linear order $<$ on X is a *well-ordering* if every nonempty subset of X has a smallest element. If $<$ is a well-ordering on X, we say X is *well-ordered* by $<$. An example of a well-ordered set is the set of natural numbers $\mathbb{N}$ with the standard ordering. It is well known (Jech, 2003, Theorem 5.1) that the Axiom of Choice is equivalent to the statement that every set can be well-ordered. Finally, if X is a well-ordered set and $\alpha \in X$, let $I_\alpha = \{x \in X : x < \alpha\}$. Such a set is called an *initial segment* of the well-ordering $<$. For example, for $n \in \mathbb{N}$ (with the standard ordering), we have $I_n = \{1, 2, \ldots, n-1\}$.

Two sets X and Y have the same *cardinality* if there is a bijection between X and Y. We write this as $|X| = |Y|$. The cardinality of X is less than or equal to Y if there is a one-to-one function $f : X \to Y$. We write this as $|X| \leq |Y|$. Finally, the cardinality of X is strictly less than Y if $|X| \leq |Y|$ but not $|X| = |Y|$. It is well-known (Jech, 2003, Theorem 3.2) that for every pair of sets X and Y we have $|X| \leq |Y|$ or $|Y| \leq |X|$, and moreover, $|X| \leq |Y|$ and $|Y| \leq |X|$ implies $|X| = |Y|$.

A set X is *finite* if $|X| < |\mathbb{N}|$ and it is *countable* if $|X| \leq |\mathbb{N}|$. The cardinality of $\mathbb{N}$ is denoted by $\aleph_0$, so that $|\mathbb{N}| = \aleph_0$. If a set X is not countable, then it is *uncountable*. For example, it is well known (Jech, 2003, Theorem 4.1) that the set of rational numbers is countable, while the set of real numbers, $\mathbb{R}$, is uncountable. The cardinality of $\mathbb{R}$ is denoted $\mathfrak{c}$ and so $\aleph_0 < \mathfrak{c}$.

It is easy to check that the relation of having the same cardinality is an equivalence relation, and therefore to each equivalence class we associate a *cardinal*. Thus, cardinals are a way of representing the size of a sets, including infinite sets. Every natural number is a cardinal and $\aleph_0$ and $\mathfrak{c}$ are other examples of cardinals. Although we do not give a formal definition of cardinals, we can think of them as generalizations of natural numbers that identify the size of infinite sets. The cardinality of the natural numbers, $\aleph_0$, is special in that it is the *least infinite cardinal*. That is, for every infinite set X, we have $\aleph_0 \leq |X|$.

Two facts about cardinals will be important to what is to come. First, every cardinal can be used to generate a strictly larger cardinal. This follows from a classic result of Cantor (Jech, 2003, Theorem 3.1) that for every set X, $|X| < |2^X|$, where 2^X denotes the power set of X. Second, cardinals are well-ordered by $<$. Therefore there is a least cardinal strictly larger than $\aleph_0$. This is the *least uncountable cardinal* and is denoted $\aleph_1$. Similarly, $\aleph_2$ is the least cardinal strictly greater than $\aleph_1$, and so on.

A natural question is whether $\mathfrak{c}$, the cardinality of $\mathbb{R}$, is the same as the least uncountable cardinal, $\aleph_1$. In fact, the statement $\mathfrak{c} = \aleph_1$ is known as the Continuum Hypothesis, abbreviated CH, and this fundamental question was famously shown to be independent of the standard axioms of set theory (including the Axiom of Choice) in the sense that it can be neither proved or disproved using the standard axioms.

More specifically, we take the standard axioms of set theory as the Zermelo-Fraenkel axiom system, including the Axiom of Choice, abbreviated as ZFC. If a proof of statement ϕ exists using the axioms of ZFC, we say ϕ is *provable in ZFC*. If $\neg\phi$ is not provable in ZFC, we say ϕ is *consistent with ZFC*. If neither ϕ or $\neg\phi$ is provable in ZFC, then we say that ϕ is *undecidable* in ZFC (or *independent* of ZFC). As originally shown by Gödel (1940) and Cohen (1963, 1964), we have the following proposition:

Proposition 1 (Gödel-Cohen). *The statement CH is undecidable in ZFC.*

We will rely on one other undecidable statement that involves the following two *cardinal characteristics of the continuum*.[4] These are the cardinalities of some specific sets of real numbers whose values are not pinned down by ZFC.

Let $\mathcal{I}$ be the collection of Lebesgue measure zero subsets of $\mathbb{R}$. The *uniformity number* of the reals is denoted by $\mathrm{uni}(\mathcal{I})$ and is defined by

$$\mathrm{uni}(\mathcal{I}) = \min\{|A| : A \notin \mathcal{I}\}.$$

That is, $\mathrm{uni}(\mathcal{I})$ is the minimal cardinality of a set that is not measure zero.[5] As every countable set of reals is measure zero and $\mathbb{R}$ is uncountable, we have $\aleph_0 < \mathrm{uni}(\mathcal{I}) \leq \mathfrak{c}$. Another common notation for the uniformity number is $\mathrm{non}(\mathcal{I})$.

The *covering number* of the reals is denoted $\mathrm{cov}(\mathcal{I})$ and is defined by

$$\mathrm{cov}(\mathcal{I}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ such that } \bigcup_{A\in\mathcal{A}} A = \mathbb{R}\right\}.$$

That is, $\mathrm{cov}(\mathcal{I})$ is the minimal cardinality of a collection of measure zero sets that cover $\mathbb{R}$.[6] As a countable union of measure zero sets is measure zero and $\mathbb{R}$ is the union of singletons, we have $\aleph_0 < \mathrm{cov}(\mathcal{I}) \leq \mathfrak{c}$.

From these observations, it should be clear that the statement $\mathrm{uni}(\mathcal{I}) = \mathrm{cov}(\mathcal{I})$ is implied by the Continuum Hypothesis and therefore by Proposition 1, this statement is consistent with ZFC. As it turns out, every possible cardinal ordering of $\mathrm{uni}(\mathcal{I})$ and $\mathrm{cov}(\mathcal{I})$ is consistent with ZFC. That is, the relationships $\mathrm{uni}(\mathcal{I}) < \mathrm{cov}(\mathcal{I})$, $\mathrm{uni}(\mathcal{I}) > \mathrm{cov}(\mathcal{I})$, and $\mathrm{uni}(\mathcal{I}) = \mathrm{cov}(\mathcal{I})$

[4]For more details, see Bartoszynski and Judah (1995); Jech (2003); Bartoszynski (2009)

[5]As cardinals are well-ordered by $<$, this minimum is obtained.

[6]Again, this minimum is obtained because cardinals are well-ordered by $<$.

are all consistent with ZFC. This fact makes up one small part of Cichoń's diagram, which summarizes all of the provable and undecidable relationships between twelve different cardinal characteristics of the continuum.[7] A concise summary of Cichoń's diagram can be found in Goldstern, Kellner and Shelah (2019), while a lengthier discussion of the methods used to prove it is contained in Blass (2009).[8] An exhaustive proof is contained in Bartoszynski and Judah (1995, Ch. 7).

The specific fact that we isolate from Cichoń's diagram is the following proposition:

Proposition 2. *The statement $\aleph_1 = uni(\mathscr{L}) < cov(\mathscr{L}) = \mathfrak{c}$ is undecidable in ZFC.*

We will rely on this fact in the proof of Theorem 1.

2.2 Ergodic Theory

Ergodic theory is the study of statistical properties of dynamic systems. A fundamental result in ergodic theory is Birkhoff's theorem, which states that for an ergodic transformation, the long-run time average of the states of the system is the same as the space average of the states. Here, an ergodic transformation is, roughly speaking, one in which repeated application on a set of positive measure eventually fills all of the space. A formal definition, as well as more discussion and proofs of Birkhoff's theorem can be found in Eisner et al. (2015, Ch. 11) or Dajani and Kalle (2021, Ch. 3).

Proposition 3 (Birkhoff's Theorem)**.** *Suppose (X, Σ, μ) is a measure space. Then a measure-preserving transformation $T : X \to X$ is ergodic if and only if for every integrable $f : X \to \mathbb{R}$ and for μ-almost every $x \in X$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(x)) = \int_X f \, d\mu.$$

The specific measure-preserving transformation we apply this theorem to is known as Irrational Rotation. For an irrational $\alpha \in (0,1)$, let $T : [0,1) \to [0,1)$ be defined by $T(x) = x + \alpha \pmod 1$. In other words, $T(x) = x + \alpha - \lfloor x + \alpha \rfloor$, where $\lfloor x \rfloor$ is the largest integer not greater than x. Applying this transformation j times gives $T^j(x) = x + j\alpha \pmod 1$.

Another well-known result in ergodic theory is that Irrational Rotation is an ergodic transformation that preserves Lebesgue measure.[9] Thus we have the following specialization

[7] This diagram was originally developed in Fremlin (1984) and Bartoszyński, Judah and Shelah (1993).
[8] See Section 11 in Blass (2009) and, in particular, Table 4.
[9] See Dajani and Kalle (2021, Sec. 2.3) or Einsiedler and Ward (2011, Sec. 2.3).

of Birkhoff's Theorem:

Corollary 1. *Fix an irrational $\alpha \in (0,1)$. For every integrable $f : [0,1] \to \mathbb{R}$ and almost every $x \in [0,1]$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(x + j\alpha \pmod 1) = \int_0^1 f(t)\, dt.$$

This result can also be seen as a consequence of the Equidistribution Theorem, which states that, for an irrational $\alpha \in (0,1)$, the sequence

$$\alpha, 2\alpha, 3\alpha, \ldots \pmod 1$$

is uniformly distributed on $[0,1]$.

3 Results

In this section, we give a precise formulation of our statement and show that it is undecidable.

For a bounded function $u : [0,1] \times [0,1] \to \mathbb{R}$, a *bounded two-player zero-sum game on the unit square given by u* is a strategic form game with two *players, strategy sets* $S_1 = S_2 = [0,1]$ and *utility functions* $u_1(s_1, s_2) = u(s_1, s_2) = -u_2(s_1, s_2)$ for all $(s_1, s_2) \in [0,1] \times [0,1]$. A *mixed strategy for player i* in such a game, denoted σ_i, is a Borel probability measure on $[0,1]$.

Consider players playing a mixed strategy profile (σ_1, σ_2). The expected payoff to player i of playing pure strategy s_i when her opponent plays a mixed strategy σ_j is given by

$$U_i(s_i \mid \sigma_j) = \int_0^1 u_i(s_i, s_j)\, d\sigma_j(s_j),$$

assuming that this integral exists. From this, we define the expected payoff to a player of the mixed strategy profile (σ_1, σ_2) by

$$\bar{U}_1(\sigma_1, \sigma_2) = \int_0^1 U_1(s_1 \mid \sigma_2)\, d\sigma_1(s_1) \quad \text{and} \quad \bar{U}_2(\sigma_1, \sigma_2) = \int_0^1 U_2(s_2 \mid \sigma_1)\, d\sigma_2(s_2)$$

assuming that both of these integrals exist.

For a function $f : [0,1] \times [0,1] \to \mathbb{R}$, we follow convention and define the function $f_x : [0,1] \to \mathbb{R}$ by $f_x(y) = f(x,y)$ and the function $f_y : [0,1] \to \mathbb{R}$ by $f_y(x) = f(x,y)$. The statement about zero-sum games that we are interested in is the following:

Statement A**:** There exists a bounded two-player zero-sum game on the unit square given by u and a profile of absolutely continuous mixed strategies $\sigma = (\sigma_1, \sigma_2)$ such that

1. for almost every $x, y \in [0, 1]$, the functions u_x and u_y are Lebesgue measurable,
2. for $i = 1, 2$ and $j \neq i$, the integral given by $U_i(s_i \mid \sigma_j)$ exists for almost every $s_i \in [0, 1]$ and $U_i(s_i \mid \sigma_j)$ is a Lebesgue measurable function of s_i.
3. $\bar{U}_1(\sigma_1, \sigma_2) + \bar{U}_2(\sigma_1, \sigma_2) \neq 0$.

In other words, this statement asks whether there is a bounded utility function u for a two-player zero-sum game and absolutely continuous mixed strategies such that the utility function is measurable in each variable, holding the other fixed and the expected utility of almost every pure strategy exists and is measurable, but the sum of the expected utilities of the two player is not zero-sum. As noted in the Introduction, from Baye, Kovenock and De Vries (2012), we know that there exists an unbounded utility function that satisfies parts (1)-(3) of this statement. On the other hand, a bounded and Lebesgue measurable utility function that satisfies part (2) must have $\bar{U}_1(\sigma_1, \sigma_2) + \bar{U}_2(\sigma_1, \sigma_2) = 0$. In this way, statement A is an intermediate question about two-player zero-sum games.

As we now show, this intermediate question is unanswerable.[10] Our main result is the following:

Theorem 1. *Statement A is undecidable in ZFC.*

The remainder of this section is devoted to a proof of this result. We begin with a lemma dealing with a specific well-ordering of $[0, 1]$. We then give two lemmas that shows that Statement A can be true or false depending on which of two undecidable statements we assume are true. These lemmas are then combined to prove Theorem 1.[11]

We begin with an initial lemma. As a consequence of the Axiom of Choice, there exists a well-ordering of the unit interval $[0, 1]$. Assuming the Continuum Hypothesis, there exists a well-ordering of $[0, 1]$ with a remarkable feature, namely that every initial segment of

[10]The assumption that players use absolutely continuous mixed strategies is made for convenience. A similar undecidable statement can be given for a broader class of mixed strategies, at the cost of additional complexity in the proof. Specifically, our arguments could be adapted to allow non-singular mixed strategies that have mass points. But see Wald (1950) and Parthasarathy (1970) for the value of restrictions on the class of mixed strategies in two-player zero-sum games.

[11]Careful inspection of these two lemmas show that Statement A remains undecidable if the quantifier "for almost every $x, y \in [0, 1]$" is replaced with "for every $x, y \in [0, 1]$."

the well-ordering is countable. While this is a standard result in the theory of ordinals, for completeness we present an elementary proof here.[12]

Lemma 1. *Suppose that CH holds. Then there exists a well-ordering $<'$ of $[0,1]$ such that for every $\alpha \in [0,1]$, the initial segment I_α is countable.*

Proof. Assume CH holds. There exists a well-ordering $<$ of $[0,1]$. If, for the well-ordering $<$, every set I_α is countable, then we are done. So suppose there is some α' such that $I_{\alpha'}$ is uncountable. Because $<$ is a well-ordering, there exists a smallest such α' according to $<$, call it β. The initial segment I_β is uncountable and it follows from CH that I_β has the cardinality of the continuum. Therefore there exists a bijection $g : [0,1] \to I_\beta$.

We can now construct the desired well-ordering of $[0,1]$. To do so, define a relation $<'$ on $[0,1]$ by $x <' y$ if and only if $g(x) < g(y)$. It is easy to check that $<'$ is a linear order. To show that it is a well-ordering, let $A \subseteq [0,1]$. We must show A has a minimal element. Consider the image $g(A)$ under the well-ordering $<$. This set has a minimal element, which is equal to $g(a^*)$ for some $a^* \in A$. As $g(a^*)$ is a minimal element of $g(A)$, $g(a^*) < g(b)$ for all $b \in A \setminus \{a^*\}$. This implies that $a^* <' b$ for all $b \in A \setminus \{a^*\}$, so a^* is a minimal element of A. Therefore $<'$ is a well-ordering of $[0,1]$.

Finally, for $\alpha \in [0,1]$, consider the set $I'_\alpha = \{y \in [0,1] : y <' \alpha\}$. By definition, $y <' \alpha$ if and only if $g(y) < g(\alpha)$, so the image $g(I'_\alpha) = I_{g(\alpha)}$. But $g(\alpha) \in I_\beta$ and therefore $g(\alpha) < \beta$. This means $I_{g(\alpha)}$ cannot be uncountable, as β is minimal. We conclude that I'_α is countable. Thus the well-ordering $<'$ has the desired feature. □

Our second lemma shows that if we assume that the Continuum Hypothesis holds, then it follows that Statement A holds. This lemma relies on a construction due to Sierpiński (1920), which was further examined by Freiling (1986).

Lemma 2. *If CH holds, then Statement A is true.*

Proof. Suppose that CH holds, so that $\mathfrak{c} = \aleph_1$. Then by Lemma 1, there is a well-ordering $<'$ of $[0,1]$ such that for every $\alpha \in [0,1]$, the initial segment I_α of $<'$ is countable.

Now, for $i = 1,2$, let $S_i = [0,1]$ and define u_i by

$$u_i(s_i, s_j) = \begin{cases} 0 & \text{if } s_i = s_j \\ +1 & \text{if } s_j <' s_i \\ -1 & \text{if } s_i <' s_j \end{cases}$$

[12]See Stein and Shakarchi (2005, p. 96).

From this definition, it is clear that $u_1(s_1, s_2) = -u_2(s_1, s_2)$ for all $s_1, s_2 \in [0,1]$, so this is a bounded zero-sum game on the unit square.

Next, we show that for every pure strategy s_i, the payoff function $u_i(s_i, s_j)$ equals -1 at all but a countable number of values of s_j. For every $s_i \in [0,1]$ we have

$$\{s_j \in [0,1] : u_i(s_i, s_j) \neq -1\} = \{s_j \in [0,1] : s_j <' s_i \text{ or } s_j = s_i\} = I_{s_i} \cup \{s_i\}.$$

The set I_{s_i} is countable and therefore $I_{s_i} \cup \{s_i\}$ is countable as well. This proves the claim. From this it follows that u_x and u_y are measurable for every $x, y \in [0,1]$. This is part (1) of Statement A.

Now fix an arbitrary profile (σ_1, σ_2) of absolutely continuous mixed strategies. Part (2) of Statement A holds as, for every $s_i \in [0,1]$

$$U_i(s_i \mid \sigma_j) = \int_0^1 u_i(s_i, s_j)\, d\sigma_j(s_j) = -1$$

because σ_j is absolutely continuous. Given this, it is immediate that

$$\bar{U}_i(\sigma_i, \sigma_j) = \int_0^1 U_i(s_i \mid \sigma_j)\, d\sigma_i(s_i) = \int_0^1 (-1)\, d\sigma_i(s_i) = -1$$

for $i = 1, 2$. Therefore $\bar{U}_1(\sigma_1, \sigma_2) + \bar{U}_2(\sigma_1, \sigma_2) = -2$, which means Statement A is true. □

It is worth noting that the proof of this lemma actually establishes the truth of a stronger statement than statement A. Specifically, it shows there exists a bounded two-player zero-sum game on the unit square given by u such that parts (1)-(3) in Statement A hold for *every* profile of absolutely continuous mixed strategies.

Our next lemma states that if the uniformity number of the reals is less than the covering number of the reals, then Statement A is true. This result is similar to a result in Shipman (1990), although we use a different method of proof. Specifically, the proof begins by showing that Statement A can be used to construct, using methods of ergodic theory, a function similar to the utility function in Lemma 2. For almost every $x \in [0,1]$, this function equals $\bar{U}_1(\sigma_1, \sigma_2)$ for almost every $y \in [0,1]$, and at the same time, for almost every $y \in [0,1]$, this function equals $-\bar{U}_2(\sigma_1, \sigma_2)$ for almost every $x \in [0,1]$. Next, we consider, for each x, the set of y such that the function equals $-\bar{U}_2(\sigma_1, \sigma_2)$. These will be measure zero sets and it turns out that, roughly speaking, for a collection of x values of cardinality $\text{uni}(\mathscr{L})$, these sets will cover $[0,1]$. This means that $\text{uni}(\mathscr{L}) \geq \text{cov}(\mathscr{L})$.

Lemma 3. *If* $\aleph_1 = uni(\mathscr{I}) < cov(\mathscr{I}) = \mathfrak{c}$ *holds, then Statement A is false.*

Proof. For a proof by contradiction, suppose $\aleph_1 = \text{uni}(\mathscr{I}) < \text{cov}(\mathscr{I}) = \mathfrak{c}$ holds but Statement A is true. That is, suppose there exists a bounded zero-sum game on the unit square and a profile $\sigma = (\sigma_1, \sigma_2)$ of absolutely continuous mixed strategies such that parts (1) and (2) of Statement A hold, but $\bar{U}_1(\sigma_1, \sigma_2) + \bar{U}_2(\sigma_1, \sigma_2) \neq 0$.

As σ_1 and σ_2 are absolutely continuous, by the Radon–Nikodym theorem there exist integrable functions g_1 and g_2 such that $\sigma_i(A) = \int_A g_i(t)\,dt$ for every Borel set $A \subseteq [0,1]$. Define $h : [0,1] \times [0,1] \to \mathbb{R}$ by

$$h(x,y) = u(x,y)g_1(x)g_2(y).$$

In addition, fix an irrational $\alpha \in (0,1)$ and define

$$w(x,y) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} h_y(x + j\alpha \pmod 1) & \text{if this limit exists} \\ 0 & \text{if not.} \end{cases}$$

and

$$v(x,y) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} w_x(y + j\alpha \pmod 1) & \text{if this limit exists} \\ 0 & \text{if not.} \end{cases}$$

By part (1) of Statement A, the function h_y is integrable for almost all y and so by Corollary 1,

$$\begin{aligned} w(x,y) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} h_y(x + j\alpha \pmod 1) &= \int_0^1 h_y(t)\,dt \\ &= \int_0^1 u(t,y)g_1(t)g_2(y)\,dt = -g_2(y)U_2(y,\sigma_1) \end{aligned}$$

for almost every x. By part (2) of Statement A, $U_2(y,\sigma_1)$ is integrable for almost every y. Then again by Corollary 1, for almost every y,

$$\begin{aligned} v(x,y) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} -g_2(y + j\alpha \pmod 1)U_2(y + j\alpha \pmod 1, \sigma_1) = \\ -\int_0^1 U_2(t,\sigma_1)g_2(t)\,dt = -\bar{U}_2(\sigma_1,\sigma_2). \end{aligned}$$

To sum up, we have shown that for almost every y, $v(x,y) = -\bar{U}_2(\sigma_1,\sigma_2)$ for almost every x. From this, define

$$B_2 = \{y \in [0,1] : v(x,y) = -\bar{U}_2(\sigma_1,\sigma_2) \text{ for almost every } x\},$$

which is a set of full measure.

Next, note that by part (1) of Statement A, for almost every x, the function $h_{x+j\alpha}$ is measurable for every j. Therefore, for almost every x, the function w_x is integrable. This means that for almost every x, by Corollary 1,

$$v(x,y) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} w_x(y + j\alpha \pmod 1) = \int_0^1 w_x(t)\, dt$$

for almost every y. From this we see that for almost every x,

$$v(x,y) = \int_0^1 w_x(t)\, dt = \int_0^1 \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} h_t(x + j\alpha \pmod 1)\, dt$$

for almost every y. Applying the Dominated Convergence Theorem, we have, for almost every x,

$$\begin{aligned} v(x,y) &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_0^1 h_t(x + j\alpha \pmod 1)\, dt \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g_1(x + j\alpha \pmod 1) \int_0^1 u(x + j\alpha \pmod 1, t) g_2(t)\, dt \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g_1(x + j\alpha \pmod 1) U_1(x + j\alpha \pmod 1, \sigma_2) \end{aligned}$$

for almost every y. Applying Corollary 1 one final time, we find that for almost every x, $v(x,y) = \int_0^1 U_1(t,\sigma_2) g_1(t)\, dt = \bar{U}_1(\sigma_1,\sigma_2)$ for almost every y. From this, define

$$B_1 = \{x \in [0,1] : v(x,y) = \bar{U}_1(\sigma_1,\sigma_2) \text{ for almost every } y\},$$

which is a set of full measure.

Finally, fix a set $M \subseteq [0,1]$ that is not measure zero with $|M| = \mathrm{uni}(\mathcal{I})$ and let $y \in B_2$

be arbitrary. From the definition of B_2, the set $E_y = \{x \in [0,1] : v(x,y) = -\bar{U}_2(\sigma_1, \sigma_2)\}$ is full measure. Since $M \cap B_1$ is not measure zero, there exists a point $\hat{x}(y) \in M \cap B_1$ such that $v(\hat{x}(y), y) = -\bar{U}_2(\sigma_1, \sigma_2)$. From this, for every $x \in M \cap B_1$ we define

$$G_x = \{y \in B_2 : v(x,y) = -\bar{U}_2(\sigma_1, \sigma_2)\}$$

and it follows that $B_2 = \bigcup_{x \in M \cap B_1} G_x$. Moreover, it follows from the definition of B_1 that G_x is measure zero for every $x \in M \cap B_1$, as $\bar{U}_1(\sigma_1, \sigma_2) \neq -\bar{U}_2(\sigma_1, \sigma_2)$. Now, for every $x \in M \cap B_1$, define $H_x = G_x \cup ([0,1] \setminus B_2)$. Clearly, H_x is the union of two measure zero sets and is thus measure zero for every $x \in M \cap B_1$. Moreover, we have

$$\bigcup_{x \in M \cap B_1} H_x = \bigcup_{x \in M \cap B_1} G_x \cup ([0,1] \setminus B_2) = [0,1].$$

This means that $\mathrm{uni}(\mathcal{L}) = |M| \geq |M \cap B_1| \geq \mathrm{cov}(\mathcal{L})$ and this contradiction completes the proof. □

We now have done all the hard work needed to prove Theorem 1. All that remains is to combine the two previous lemmas and Propositions 1 and 2.

Proof of Theorem 1. We must show that neither statement A or its negation is provable in ZFC. First, suppose that statement A is provable in ZFC. Then by the contrapositive of Lemma 3, the statement $\aleph_1 = \mathrm{uni}(\mathcal{L}) < \mathrm{cov}(\mathcal{L}) = \mathfrak{c}$ is provably false in ZFC. But this contradicts Proposition 2. Second, suppose that statement A is provably false in ZFC. Then by the contrapositive of Lemma 2, the Continuum Hypothesis is provably true in ZFC. But this contradicts Proposition 1. This proves that statement A is undecidable in ZFC. □

4 Conclusion

In this paper, we have shown that a particular statement about two-player zero-sum games is undecidable. Of course, this leaves open the question of what other game-theoretic statements are undecidable. In the context of two-player zero-sum games on the unit square, it seems natural to consider if it could be the case that it is undecidable whether a Minimax Theorem holds, for instance. More generally, there may be conditions under which the existence of a Nash equilibrium with certain properties is undecidable. We leave such questions for future research.

References

Bartoszynski, Tomek. 2009. Invariants of measure and category. In *Handbook of Set Theory*. Springer pp. 491–555.

Bartoszynski, Tomek and Haim Judah. 1995. *Set Theory: On the Structure of the Real Line*. CRC Press.

Bartoszyński, Tomek, Haim Judah and Saharon Shelah. 1993. "The Cichoń diagram." *The Journal of Symbolic Logic* 58(2):401–423.

Baye, Michael R, Dan Kovenock and Casper G De Vries. 2012. "The Herodotus paradox." *Games and Economic Behavior* 74(1):399–406.

Berge, Claude. 1957. Topological games with perfect information. In *Contributions to the Theory of Games, Volume 3*, ed. Albert William Tucker Melvin Dresher and Philip Wolfe. Number 39 *in* "Annals of Mathematics Studies" Princeton University Press Princeton pp. 165–178.

Blass, Andreas. 2009. Combinatorial cardinal characteristics of the continuum. In *Handbook of Set Theory*, ed. Matthew Foreman and Akihiro Kanamori. Springer pp. 395–489.

Ciesielski, Krzysztof. 1997. *Set Theory for the Working Mathematician*. Number 39 *in* "London Mathematical Society Student Texts" Cambridge University Press.

Cohen, Paul J. 1963. "The independence of the continuum hypothesis." *Proceedings of the National Academy of Sciences* 50(6):1143–1148.

Cohen, Paul J. 1964. "The independence of the continuum hypothesis, II." *Proceedings of the National Academy of Sciences* 51(1):105–110.

Dajani, Karma and Charlene Kalle. 2021. *A First Course in Ergodic Theory*. Chapman & Hall / CRC.

Einsiedler, Manfred and Thomas Ward. 2011. *Ergodic Theory: with a view towards Number Theory*. Springer-Verlag London.

Eisner, Tanja, Bálint Farkas, Markus Haase and Rainer Nagel. 2015. *Operator Theoretic Aspects of Ergodic Theory*. Springer International Publishing.

Freiling, Chris. 1986. "Axioms of symmetry: throwing darts at the real number line." *The Journal of Symbolic Logic* 51(1):190–200.

Fremlin, D. H. 1984. "Cichoń's diagram." *Publ. Math. Univ. Pierre Marie Curie* 66:1–13.

Friedman, Harvey. 1980. "A consistent Fubini-Tonelli theorem for nonmeasurable functions." *Illinois Journal of Mathematics* 24(3):390–395.

Gödel, Kurt. 1940. *The consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory*. Number 3 *in* "Annals of Mathematics Studies" Princeton University Press.

Goldstern, Martin, Jakob Kellner and Saharon Shelah. 2019. "Cichoń's maximum." *Annals of Mathematics* 190(1):113–143.

Jech, Thomas. 2003. *Set theory*. Springer.

Kechris, Alexander S. 1995. *Classical descriptive set theory: Graduate texts in mathematics*. Springer (New York, NY).

Litak, Tadeusz. 2018. "Infinite populations, choice and determinacy." *Studia Logica* 106(5):969–999.

Parthasarathy, T. 1970. "On games over the unit square." *SIAM Journal on Applied Mathematics* 19(2):473–476.

Prikry, Karel and William D Sudderth. 2016. "Measurability of the value of a parametrized game." *International Journal of Game Theory* 45:675–683.

Rabin, Michael O. 1957. Effective computability of winning strategies. In *Contributions to the Theory of Games, Volume 3*, ed. Albert William Tucker Melvin Dresher and Philip Wolfe. Number 39 *in* "Annals of Mathematics Studies" Princeton University Press Princeton pp. 147–157.

Recław, Ireneusz and Piotr Zakrzewski. 1999. "Fubini properties of ideals." *Real Analysis Exchange* pp. 565–578.

Rubinstein, Ariel. 1998. *Modeling Bounded Rationality*. MIT press.

Shipman, Joseph. 1990. "Cardinal conditions for strong Fubini theorems." *Transactions of the American Mathematical Society* 321(2):465–481.

Sierpiński, Wacław. 1920. "Sur les rapports entre l'existence des intégrales $\int_0^1 f(x,y)dx$, $\int_0^1 f(x,y)dy$ et $\int_0^1 dx \int_0^1 f(x,y)dy$." *Fundamenta Mathematicae* 1(1):142–147.

Stein, Elias M and Rami Shakarchi. 2005. *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*. Princeton University Press.

Velupillai, K Vela. 2009. "Uncomputability and undecidability in economic theory." *Applied Mathematics and Computation* 215(4):1404–1416.

Wald, Abraham. 1950. "Note on zero sum two person games." *Annals of mathematics* pp. 739–742.