Repeated Downsian Electoral Competition

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Abstract

We analyze an infinitely repeated version of the Downsian model of elections. The folk theorem suggests that a wide range of policy paths can be supported by subgame perfect equilibria when parties and voters are sufficiently patient. We go beyond this result by imposing several suitable refinements and by giving separate weak conditions on the patience of voters and the patience of parties under which every policy path can be supported. On the other hand, we show that only majority undominated policy paths can be supported in equilibrium for arbitrarily low voter discount factors: if the core is empty, the generic case in multiple dimensions, voter impatience leads us back to the problem of non-existence of equilibrium. We extend this result to give conditions under which core equivalence holds for a non-trivial range of voter and party discount factors, providing a game-theoretic version of the Median Voter Theorem in a model of repeated Downsian elections.

1 Introduction

The Median Voter Theorem of Black (1958) establishes that if voters have single-peaked preferences over a one-dimensional set of alternatives, then the median of the distribution of voter ideal points is majority-preferred to all other alternatives. In multidimensional policy spaces, however, such "core points" typically do not exist (Plott, 1967; Rubinstein, 1979; Schofield, 1983; Cox, 1984; Le Breton, 1987; Banks, 1995). The standard game-theoretic model of two-party spatial competition, examined by Downs (1957), assumes that parties are office-motivated, that they can commit to campaign platforms, and that voters eliminate weakly dominated strategies. In this setting, it is easy to show that a choice of policy by the parties is a Nash equilibrium if and only if both parties locate at a core point. Thus, the median is the unique equilibrium in the one-dimensional model, but electoral equilibria typically fail to exist in multiple dimensions. These conclusions rely, however, on the implicit assumption that the election is one-time only; there are no future contests to consider.

In this paper, we investigate the consequences of repeating this Downsian game and allowing parties and voters to anticipate the effects of their actions on future elections in a fully rational manner. Drawing on the theory of infinitely repeated games, it is a folk theorem that if players have sufficiently high discount factors, then the set of subgame perfect equilibrium outcomes of a repeated game can be sizeable (Fudenberg and Maskin, 1986). Motivated by this result, we consider the following questions concerning infinitely repeated two-party competition. Does the median remain the unique equilibrium outcome in one dimension? Do equilibria exist in multiple dimensions when the core is empty? How large does the set of equilibrium outcomes become? How do the equilibrium outcomes depend on the patience of the parties and voters?

We first show that under modest conditions, not only do electoral equilibria exist in any number of dimensions, but *every* possible sequence of policies is supportable by a subgame perfect equilibrium. We give two sets of conditions under which this stark conclusion of multiplicity holds. The first requires only that *voters* place more weight on the future than the present (discount factors greater than one half) and imposes no restriction on the parties. The second requires that *parties* place more weight on the future than the present and states weaker conditions on voter discount factors. An implication of the second result is that when the core is nonempty, every policy path can be supported in equilibrium merely if voter discount factors are positive. The availability of a core point (or policies close to being in the core) is used in our construction to facilitate the punishment of parties, so that a nonempty core can actually exacerbate the problem of multiplicity.

These results go beyond the standard folk theorem in several ways. First, they do not rely on arbitrarily patient players, but rather they describe what outcomes are supportable for a wide range of discount factors, which we allow to vary across voters and parties separately. Second, we show that arbitrary paths of policies over time, and not just expected payoffs, can be supported in equilibrium. Third, we use several suitable refinements of subgame perfect equilibrium that restrict the kinds of punishments available to parties and voters. In particular, we exclude equilibria in which voters or parties condition on how particular voters voted in the past, or even on total vote tallies in previous elections.

Furthermore, we suppose each voter acts as though pivotal in every election, essentially eliminating weakly dominated strategies in every period in the spirit of the one-shot Downsian model. Finally, we restrict ourselves to equilibria in which any voter, when indifferent concerning which party wins, flips a fair coin to decide his/her vote, treating the parties symmetrically.

With these results in hand, we next investigate the implications for equilibria when voters are impatient. We show that only policy paths close to being in the core can be supported in equilibrium as voters' discount factors go to zero, with the implication that only paths in the core can be supported for arbitrarily low voter discount factors. If the core is empty, the generic case in multiple dimensions, then voter impatience leads us back to equilibrium non-existence. We then extend this result to give a range of discount factors in which parties must choose core points (the median, in one dimension) in equilibrium, a phenomenon we refer to as "core equivalence." In this vein, we prove that when the number of voters is odd, utilities are quadratic, and voters and parties are relatively impatient, core equivalence holds.

Repeated elections have also been considered in the literature on electoral accountability, which drops the commitment assumption and modifies other details of the Downsian model, such as adding asymmetric information of one form or another.[1] While many of these models assume a single "representative" voter, Duggan (2000) and Bernhardt, Dubey, and Hughson (2004) explicitly allow for a continuum of voters, and the former paper contains simulation results suggesting core equivalence as voters become arbitrarily patient. Aragones and Postlewaite (2000) consider a related model but assume complete information. While the latter papers assume a one-dimensional policy space, Banks and Duggan (2006a) prove existence of equilibria in multiple dimensions and give analytic results on core equivalence. Work on electoral accountability differs from ours not only in removing the commitment assumption and adding asymmetric information, but also in focusing on stationary equilibria.[2]

Kramer (1977) takes a different approach by assuming office-motivated parties that can commit to policy platforms before elections and allowing for multiple dimensions. His model differs from ours, however, in that only the party out of power may choose a platform, while the incumbent party is fixed at its previous position, and in that the parties optimize myopically. Alesina (1988) takes yet another approach by assuming policy-motivated parties that cannot commit to policy platforms, assuming probabilistic voting, and considering a specific class of non-stationary equilibria, namely, those using "Nash reversion" punishments. Finally, McKelvey and Ordeshook (1985) and Shotts (2006) are examples of models focusing on the informational aspects of repeated elections with private information.

The organization of the paper is as follows. In Section 2, we describe the repeated elections framework. In Section 3, we present our equilibrium analysis. In Section 4, we end with some brief concluding remarks. An appendix contains proofs of all of our theorems, including diagrams of equilibrium constructions used in the proofs of Theorems 1 and 2.

[1]For an expository review of this literature, see Fearon (1999).

[2]As we show, in the repeated Downsian electoral model, stationarity implies core equivalence and, therefore, equilibrium existence problems in multiple dimensions.

2 The Repeated Elections Framework

2.1 The Model

The players in our model are two parties, labeled A and B, and n voters, who participate in an infinite sequence of elections. In each election, the parties simultaneously choose policy platforms from some set X of policy alternatives, generically denoted x, x', etc. At this point, we impose no structure on the set X, allowing it to be finite or infinite, perhaps a subset of the real line or a subset of multidimensional Euclidean space. We use y to denote a platform choice by party A and z to denote a choice by B. In each period, once the parties have selected platforms, the voters observe these choices and simultaneously cast ballots for A or B. In every period, the election is determined by plurality rule, with the party receiving the most votes implementing its platform for that period. In the event of a tie, parties A and B each win with probability one half, i.e., the winner is decided by the toss of a fair coin. We denote a generic policy in period t by x_t, and we let $\mathbf{x} = (x_1, x_2, \ldots)$ denote an infinite path of policies.

A *complete history* of length t, denoted h_t, is a list of the actions of all players in periods $1, 2, \ldots, t$, so that it lists the platforms of the parties, the votes of the voters in each period, and, in case of electoral ties, the outcomes of coin flips to break ties. A *partial history* of length t is a complete history of length $t-1$ together with the platforms of the parties (but not the votes of the voters) in period t. We define the *initial history,* denoted $h_0 = \emptyset$, as the empty list that describes the game at the beginning of period 1. A *complete history* is a complete history of finite length; a *partial history* is a partial history of finite length; and an *infinite history* is a list of platforms, votes, and (if applicable) coin tosses for every period. We denote the set of all complete histories of length t by H_t and the set of all complete histories by $H = \bigcup_t H_t$. A *strategy of a party* $P \in \{A, B\}$ is a mapping $\sigma_P : H \rightarrow X$, indicating the platform the party will adopt after different complete histories. Though our focus is on pure strategies, it is important to allow for randomized voting strategies when voters are indifferent. Thus, a *strategy of a voter* i is a mapping $\sigma_i : H \times X \times X \rightarrow [0, 1]$ which gives the probability of a vote for party A for each complete history and platform pair (y, z), i.e., each partial history. A profile of strategies is $\sigma = (\sigma_A, \sigma_B, \sigma_1, \ldots, \sigma_n)$.

An *electoral outcome* in period t is (i) the platforms, y_t and z_t, chosen by the parties, and (ii) the winner of the election in t (possibly determined by a tie-breaking coin flip), denoted $w_t \in \{0, 1\}$, where $w_t = 1$ indicates a win for A and $w_t = 0$ indicates a win for B. Given a complete history h_t, let

$$o(h_t) \quad = \quad ((w_1, y_1, z_1), (w_2, y_2, z_2), \ldots, (w_t, y_t, z_t))$$

denote the sequence of electoral outcomes associated with h_t. A strategy profile σ determines a distribution on infinite sequences of electoral outcomes, where randomness may be introduced by mixed voting strategies and tied elections. This distribution, in turn, determines a distribution on infinite paths of implemented policies.

We assume each voter i has a bounded utility function $u_i \colon X \rightarrow \mathbb{R}$ that reflects the voter's preferences over policies in any period. Let $\overline{u}_i$ and $\underline{u}_i$ denote upper and lower bounds, respectively, for the voter's utility function.[3] Write $x \; M \; y$ if x is *majority-preferred* to y, i.e., if the number of voters with $u_i(x) > u_i(y)$ is greater than $n/2$. Define the *core,* denoted K, as the set of majority undominated policies, i.e,

$$K \;\; = \;\; \{x \in X \mid y \; M \; x \text{ for no } y \in X\}.$$

When voter preferences are single-peaked and $X \subseteq \mathbb{R}$, it is well-known that K consists of the *median* policies, i.e., $x \in K$ if and only if the number of voters with ideal points to either side of x is less than or equal to $n/2$. We also define the related relation M°, as $x \; M^\circ \; y$ if and only if the number of voters with $u_i(x) > u_i(y)$ is greater than or equal to $n/2$. This relation can violate asymmetry and is not generally equivalent to majority preference, but the distinction disappears when the number of voters is odd. We define the *subcore,* denoted K°, as the set of M°-undominated policies, i.e.,

$$K^\circ \;\; = \;\; \{x \in X \mid y \; M^\circ \; x \text{ for no } y \in X\}.$$

Write $x \; M^* \; y$ if x is *plurality-preferred* to y, i.e., if the number of voters with $u_i(x) > u_i(y)$ is greater than the number of voters with $u_i(y) > u_i(x)$. Define the *plurality core,* denoted K^*, as the set of plurality undominated policies, i.e.,

$$K^* \;\; = \;\; \{x \in X \mid y \; M^* \; x \text{ for no } y \in X\}.$$

It is clear that $K^* \cup K^\circ \subseteq K$, that K^* and K° are not generally nested, and that $K^\circ = K$ when n is odd.

We say the core K is *strong* if it consists of one policy, say x^*, and this policy is majority preferred to all others: for all $x \in X \setminus \{x^*\}$, we have $x^* \; M \; x$. The core is necessarily strong if it is nonempty, n is odd, and voter preferences are linear.[4] As well, the core is strong if it is nonempty, n is odd, X is a convex subset of Euclidean space, and voter preferences are strictly quasi-concave (as in the standard spatial model of politics). Clearly, when the core is strong, the above concepts coincide, so that $K^* = K^\circ = K$.

Voters in our model are fully rational in that they consider the effect of their current vote on future elections in deciding how to vote. We assume that, in order to evaluate these effects, voter preferences over infinite histories are represented by the discounted sum of utilities from policies over time, i.e.,

$$(1 - \delta_i) \sum_{t=1}^{\infty} \delta_i^{t-1} [w_t u_i(y_t) + (1 - w_t) u_i(z_t)],$$

where $\delta_i \in [0, 1)$ is the discount factor for voter i. Preferences over lotteries on outcome paths are given by the expected discounted sum of utilities. Party P receives a payoff of one when it wins, zero otherwise. Thus, we assume the parties are probability of winning maximizers. Each has a discount rate $\delta_P \in [0, 1)$, and we assume A's preferences over infinite histories are given by

$$(1 - \delta_A) \sum_{t=1}^{\infty} \delta_A^{t-1} w_t$$

[3] It should be noted that the assumption that utilities are bounded is not needed for the results in section 3.1.

[4] We say preferences are *linear* if $u_i(x) = u_i(x')$ implies $x = x'$. This case is of interest when X is finite. Linearity is implicitly assumed in the literature on tournaments and voting in agendas. See Moulin (1986).

and B's preferences by

$$(1-\delta_B)\sum_{t=1}^{\infty}\delta_B^{t-1}(1-w_t).$$

Again, preferences over lotteries on outcome paths are given by expected discounted payoffs.

A specification of strategies for parties and voters is a *subgame perfect equilibrium* if it satisfies the following: no party or voter has a different strategy that, following some history, yields a distribution over outcome paths with a higher expected discounted sum of utilities. We say a subgame perfect equilibrium *supports* a policy path $\mathbf{x}$ if the distribution on infinite policy paths determined by the equilibrium strategies puts probability one on $\mathbf{x}$. Note that this can happen in one of two ways: in any period t, either both candidates adopt platform x_t or only party P adopts x_t, and it wins with probability one. If there is a subgame perfect equilibrium that supports $\mathbf{x}$, then we say the path is *supportable.*

2.2 Equilibrium Refinements

In this section, we present some restrictions on strategies in order to rule out especially implausible equilibria of the game. It is well-known that in infinitely repeated games with sufficiently patient players, a large set of outcomes can typically be supported by subgame perfect equilibria (Fudenberg and Maskin (1986)). The standard folk theorem places no limitations, however, on the types of punishments that can be used by the equilibrium supporting a particular outcome. In the context of repeated elections, we want to exclude equilibria that are less compelling on the grounds of realism, such as those in which one voter is singled out for voting the wrong way and punished in the future by parties and other voters. Thus, we focus on equilibria in which the choices of voters and parties in any period are conditioned only on previous electoral outcomes. We refer to this restriction as "outcome stationarity."

Definition 1 (OS) *A strategy profile σ satisfies* outcome stationarity *if for all t and all complete histories h_t and h'_t such that $o(h_t) = o(h'_t)$,*

1. *for each party P, $\sigma_P(h_t) = \sigma_P(h'_t)$, and*
2. *for all (y,z) and all i, $\sigma_i(h_t,y,z) = \sigma_i(h'_t,y,z)$.*

In other words, outcome stationarity requires that after any two histories with identical sequences of outcomes, the specified platform choice of each party is the same and the choices of the voters can only be conditioned on the parties' choices of platforms in the current period. Thus, following a complete history h_t and platform choices of the parties, y and z, each voter i can calculate the expected discounted sum of utilities if A is elected and if B is elected in the current period, given the strategies of the other players. Denote these continuation values by $v_i(h_t,y,z,1)$ and $v_i(h_t,y,z,0)$, respectively.

Even restricting the available strategies to those that satisfy outcome stationarity, it is trivial to establish a folk theorem-like result. In fact, the result is much stronger than the standard folk theorem because it holds for arbitrary discount factors, not just a particular range of values.

Proposition 1 *If $n \geq 3$, then every policy path is supportable by a subgame perfect equilibrium satisfying (OS).*

The proof of this proposition is straightforward. Let $\mathbf{x}$ be any given policy path. Specify strategies to that both parties choose x_t in period t, regardless of history. For the voters, as long as both parties choose the prescribed platforms in each period, the voters randomize their vote. If a party deviates from x_t, then in that and all later periods, all voters vote for the non-deviating party. Clearly, no player can gain by deviating. Subgame perfection holds because (as $n \geq 3$) it is a Nash equilibrium in the voting subgame for all voters to vote for A, regardless of their preferences over candidate platforms; similarly, it is Nash for all to vote for B.

The strategy profile specified above is clearly a subgame perfect equilibrium and satisfies outcome stationarity, but it requires some voters to vote against their preferred party. The standard response to this in a one-shot model is to eliminate weakly dominated strategies. We follow this by assuming that each voter, while taking the strategies of all players in the future as fixed, eliminates weakly dominated strategies in every voting subgame. In other words, when the continuation value to player i of having A elected in the current period is strictly higher than the continuation value of electing B, voter i votes for A, and similarly for B. This is equivalent to requiring that all voters act as if they were pivotal in the current period. Following Baron and Kalai (1993), we refer to this restriction as "stage game weak dominance."

Definition 2 (WD) *A strategy profile σ satisfies* stage game weak dominance *if for every complete history h_t and all platforms y and z,*

1. $(1-\delta_i)u_i(y) + \delta_i v_i(h_t, y, z, 1) > (1-\delta_i)u_i(z) + \delta_i v_i(h_t, y, z, 0)$ *implies* $\sigma_i(h_t, y, z) = 1$, *and*
2. *the reverse inequality implies* $\sigma_i(h_t, y, z) = 0$.

Stage game weak dominance requires voters with a strict preference to act accordingly, but it does not restrict the actions of indifferent voters. While it is true that the choice is irrelevant to such a voter, it can dramatically affect the choices of the parties. Now assuming that parties are relatively patient, it is still trivial to support arbitrary policy paths with equilibria satisfying outcome stationarity and stage game weak dominance.

Proposition 2 *Let $\delta_P \geq 1/2$ for both parties. If $K^* \neq \emptyset$, then every policy path is supportable by a subgame perfect equilibrium satisfying (OS) and (WD).*

Let $\mathbf{x}$ be any policy path. Specify strategies so that both parties choose x_t in period t and all voters flip fair coins to decide their votes, until a party, say A, deviates. Then in all future periods, both parties locate at a pre-specified plurality core policy, where each voter votes for B whenever the expected discounted payoff from electing B is weakly higher than from A. Along the equilibrium path, all voters are indifferent between the two parties, and so it is a best response for the voters randomize, and voting strategies are by definition best responses after a party deviates. Along the equilibrium path, each party receives a payoff of one half. If A deviates to platform y such that yM^*x_t, then A's payoff may increase in period t. Following that, however, a majority of voters will vote for B, and A loses with probability one thereafter. Since $\delta_A \geq 1/2$, this deviation is not profitable. Note that nonemptiness of the core can facilitate the construction of equilibria by providing a means of punishing parties off the equilibrium path, a fact that we use later.

The "Nash reversion" equilibrium used in the argument for Proposition 2 satisfies outcome stationarity and stage game weak dominance, but it depends critically on the possibility that the parties are treated asymmetrically, even when they adopt identical platforms (and are expected to always do so in the future). We therefore impose a last restriction, augmenting stage game weak dominance, which we call "party symmetry." Party symmetry requires that when indifferent between the two parties, a voter flips a fair coin to decide. Note that, as a consequence, if the inequality in Definition 2 holds for a plurality of voters after some partial history, then party A wins with probability greater than one half.

Definition 3 (PS) *A strategy profile σ satisfies* party symmetry *if for every complete history h_t, and all platforms y and z, $(1-\delta_i)u_i(y) + \delta_i v_i(h_t, y, z, 1) = (1-\delta_i)u_i(z) + \delta_i v_i(h_t, y, z, 0)$ implies $\sigma_i(h_t, y, z) = 1/2$.*

An alternative to outcome stationarity is a strengthening, called "strong stationarity," that requires parties to use strategies that are history-independent in each period and voters to condition only on the platforms of the parties in the each period.

Definition 4 (SS) *A strategy profile σ satisfies* strong stationarity *if for all t and all complete histories h_t and h'_t,*

1. *for each party P, $\sigma_P(h_t) = \sigma_P(h'_t)$, and*
2. *for all (y, z) and all i, $\sigma_i(h_t, y, z) = \sigma_i(h'_t, y, z)$.*

The next proposition shows that this strengthening of outcome stationarity is largely uninteresting, as it brings us back to the Downsian core equivalence result.[5] Note that the "only if" direction in the following proposition does not rely on party symmetry because strong stationarity and weak dominance are enough to imply that all voters must vote sincerely in every period.

Proposition 3 *If a policy path $\mathbf{x}$ is supportable by a subgame perfect equilibrium satisfying (SS) and (WD), then $x_t \in K^*$ for all t. Conversely, if $x_t \in K^*$ for all t, then $\mathbf{x}$ is supportable by a subgame perfect equilibrium satisfying (SS), (WD), and (PS).*

[5] This proposition is similar to the result that the only Markov perfect equilibrium of an infinitely repeated game is the infinite repetition of a stage game equilibrium.

In a subgame perfect equilibrium satisfying strong stationarity, it is clear that continuation values $v_i(h_t, y, z, P)$ are independent of h_t, y, z, and P: regardless of their values, the parties will each choose some y and z in period $t+1$, and in every period thereafter, and any voter who conditions only on those platforms will vote the same way in every period. Similarly, the parties' expected discounted payoffs are constant across all histories. If policy path $\mathbf{x}$ is supported by such an equilibrium and $x_t \notin K^*$ for some t, then some party locates at x_t and wins with probability at least one half. Then the other party, say A, can locate at y such that $yM^*z = x_t$. Since continuation values are fixed, the plurality of voters who strictly prefer y must, by weak dominance, vote for A, who therefore wins with probability greater than one half. By stationarity, A's continuation value does not change, and the deviation is profitable, a contradiction. It is straightforward to construct equilibria supporting paths in the plurality core for the converse direction of the proposition.

3 Equilibrium Analysis

3.1 Multiplicity of Equilibria

In this subsection, we show that subgame perfection, even augmented by our refinements, is completely unrestrictive under two sets of conditions, each using a different logic to support policy paths. Our first result focuses on voter patience, showing that if voters place more weight on the future than on the present, then every path of policies can be supported in equilibrium. This strengthens the usual conclusion of the folk theorem, which is stated in terms of payoffs rather than outcomes, while at the same time imposing relatively weak restrictions on discount factors: rather than assume that all players are arbitrarily patient, we only restrict the patience of voters, and it is sufficient that voter discount factors exceed one half. Furthermore, we support policy paths with equilibria satisfying a variety of refinements, discussed in the previous section.

Theorem 1 *Let $\delta_i > 1/2$ for all voters. Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

The equilibrium constructed in the appendix can be described roughly as follows. We specify that both parties choose platform x_t in period t, unless some party has deviated. Along the equilibrium path, voters are indifferent between the parties and so flip coins to decide their ballots, giving the parties expected discounted equilibrium payoffs of one half. In any period where one party, say A, has deviated to $y \neq x_t$, future policy platforms depend on several factors. If $x_t \; M^* \; y$, then the deviation is ignored and the parties "return to the equilibrium path." If $y \; M^* \; x_t$, then future policy platforms depend on which party wins: if the deviating party, A, wins, then the parties adopt x_t in all future periods; if party B wins, then the parties switch to the deviant platform, y, in all future periods. If A deviates to y such that $y \; M^* \; x_t$, then voter i votes for A if $u_i(x_t) > u_i(y)$, votes for B if the opposite inequality holds, and flips a coin if equality holds. Given the strategies of the parties, and given that $\delta_i > 1/2$, these voting strategies are best responses satisfying stage game weak dominance and party symmetry. Given the strategies of the voters, no party has an incentive to deviate: if A deviates to y such that $y \; M^* \; x_t$, for example, then a plurality of voters will vote for B, so A wins with probability less than one half in period t (and will win with probability one half in the future), giving it an expected discounted payoff less than one half.

We can obtain the result of Theorem 1 with even weaker conditions on voter discount factors by assuming parties are relatively patient. The next theorem assumes that party discount factors are at least one half and shows that if the discount factors of some voters exceed a certain level, then every path of policies can be supported in equilibrium. The critical discount factor level depends on the preferences of voters. If the core is nonempty, for example, then it is essentially zero: as long as all voters put positive weight on the future, every policy path is supportable.

Theorem 2 *Let $\delta_P \geq 1/2$ for both parties. Assume there exists a policy x^* such that (i) $x^* \; M \; x^0$ for some policy x^0, and (ii) for every policy x', there exist a majority coalition C and policies r and s such that for all voters $i \in C$, we have $u_i(r) \geq u_i(s)$ and, in case $u_i(x') \geq u_i(x^*)$, we also have*

$$\frac{u_i(x') - u_i(x^*)}{u_i(r) - u_i(s) + u_i(x') - u_i(x^*)} \quad < \quad \delta_i.$$

Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).

The equilibrium constructed for Theorem 2 uses some of the same ideas as that for Theorem 1, but it is more complex, so we leave the reader to the formal description in the appendix. One difference between the constructions is of note. To prove Theorem 1, we specified party strategies so that the parties always choose the same platform and, unless one party deviated, voters treat the parties symmetrically. In proving Theorem 2, we also specify that parties always choose the same platform, but in some subgames off the equilibrium path, one party wins with probability one. This is consistent with voter incentives because the parties' current platforms are identical and, by design, the expectations about the future are worse for a majority of voters if the out-party wins. A difficulty that arises is the potential for the out-party to deviate profitably. Because we seek to use the weakest possible assumptions on voters' discount factors, we employ the incentives of relatively patient parties and make use of the policy x^* satisfying conditions (i) and (ii) of the theorem. It is in this role that a subcore policy, if one exists, can facilitate the construction of punishments off the equilibrium path.

Note that the demands on voter discount factors in Theorem 2 are indeed weaker than in Theorem 1: if voter discount factors exceed one half, then x^* can be specified as any policy that is not bottom-ranked according to majority preferences, and r and s can be set to x' and x, respectively, to fulfill the conditions of the Theorem 2. Alternatively, when the subcore is nonempty, we may specify x^* as any element of that set that is not bottom-ranked. Thus, Theorem 2 shows that the result of Proposition 2 can be obtained even if party symmetry is imposed.

We can use Theorem 2 to express the critical discount factor level for voters in terms of a measure of how close the subcore is to being nonempty. Let $\mathcal{M}$ denote the collection of all majority coalitions, and define

$$\psi^\circ(x) \;=\; \sup_{x' \in X} \min_{C \in \mathcal{M}} \max_{i \in C} u_i(x') - u_i(x).$$

Note that ψ is non-negative and that $\psi^\circ(x) = 0$ if and only if $x \in K^\circ$. Intuitively, ψ° measures how far the policy x is from being in the subcore, given the utility functions $u_1, \ldots, u_n$. Accordingly, $\inf_{x \in X} \psi^\circ(x)$ measures how far voter utilities are from admitting a non-empty subcore. Also, let

$$\Delta \;=\; \min_{C \in \mathcal{M}} \sup_{r,s \in X} \min_{i \in C} u_i(r) - u_i(s)$$

measure the possible incentives available to influence the most resistant majority coalition. The following corollary of Theorem 2 provides a critical discount factor level for voters in terms of these concepts.

Corollary 1 *Let $\delta_P \geq 1/2$ for both parties. Assume there exists a policy x^* such that (i) $x^* \; M \; x^0$ for some policy x^0, and (ii) for all voters i we have*

$$\frac{\psi^\circ(x^*)}{\Delta + \psi^\circ(x^*)} \;<\; \delta_i.$$

Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).

The lefthand side of the inequality in the corollary is increasing in $\psi^\circ(x^*)$. Therefore, the condition of the corollary implies that for all voters i and all policies x',

$$\frac{\min_{C \in \mathcal{M}} \max_{j \in C} u_j(x') - u_j(x^*)}{\Delta + \min_{C \in \mathcal{M}} \max_{j \in C} u_j(x') - u_j(x^*)} \;<\; \delta_i.$$

This in turn implies that for all voters i and all policies x', there exists a majority C such that

$$\frac{\max_{j \in C} u_j(x') - u_j(x^*)}{\Delta + \max_{j \in C} u_j(x') - u_j(x^*)} \;<\; \delta_i.$$

Since the lefthand side is decreasing in Δ, it follows by construction that for all voters i and all policies x', there exist a majority, without loss of generality, C itself, and policies r and s such that

$$\frac{\max_{j \in C} u_j(x') - u_j(x^*)}{(\min_{j \in C} u_j(r) - u_j(s)) + (\max_{j \in C} u_j(x') - u_j(x^*))} \;<\; \delta_i.$$

Choosing any voter $i \in C$, this yields the inequality in Theorem 2, and the conclusion of the corollary follows.

Note that the existence of a policy satisfying condition (ii) of Corollary 1 can be written more succinctly in the form

$$\frac{\inf_{x \in X} \psi^\circ(x)}{\Delta + \inf_{x \in X} \psi^\circ(x)} \;<\; \delta_i,$$

which uses our measure of how close the subcore is to being nonempty. To fulfill the condition of Theorem 2, it must be that the infimum $\inf_{x \in X} \psi^{\circ}(x)$ can be approximated by a policy that satisfies condition (i), i.e., it is majority preferred to at least one other policy, necessitating the more complicated statement of the corollary. This added requirement is quite weak, however, and by far the greater demand of the corollary is made in condition (ii).

A further implication is immediate: if the subcore contains a policy x^*, then $\psi^{\circ}(x^*) = 0$, and Corollary 1 implies that, assuming party discount factors are at least one half, every policy path is supportable in equilibrium as long as x^* is majority preferred to at least one other policy and voter discount factors are strictly positive.

Corollary 2 *Let $\delta_P \geq 1/2$ for both parties, and assume there exists a policy $x^* \in K^{\circ}$ such that $x^*\ M\ x^0$ for some policy x^0. If $\delta_i > 0$ for all voters, then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

The lefthand side of the inequality in Corollary 1 goes to zero with $\psi^{\circ}(x^*)$. Thus, it becomes easier to fulfill condition (ii) and to support policy paths when the subcore is closer to nonempty. As well, the lefthand side is decreasing in the available incentives Δ, so that, as we would expect, it becomes easier to support paths when stronger incentives are available. The next corollary uses the former observation to state a last implication of Theorem 2. We use the notation $K(u_1, \ldots, u_n)$ to denote the core generated by the utilities $u_1, \ldots, u_n$, and let $\Delta(u_1, \ldots, u_n)$ denote the corresponding available incentives.

Corollary 3 *Let $\delta_P \geq 1/2$ for both parties, and let $\delta_i > 0$ for all voters. Assume that X is a compact topological space, that u_i is continuous for all voters, that $K(u_1, \ldots, u_n)$ is nonempty and strong, and that $\Delta(u_1, \ldots, u_n) > 0$. Let $\{(u_1^m, \ldots, u_n^m)\}$ be a sequence of profiles of continuous utility functions converging uniformly to $(u_1, \ldots, u_n)$. Then, for high enough m, every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

To see this result, let $K(u_1, \ldots, u_n) = \{x^*\}$. Define the mapping ϕ from policy pairs and profiles of continuous utility functions by

$$\phi(x, x', \hat{u}_1, \ldots, \hat{u}_n) \quad = \quad \min_{C \in \mathcal{M}} \max_{i \in C} \hat{u}_i(x') - \hat{u}_i(x),$$

where we topologize the space of utility profiles by the sup norm. Then ϕ is jointly continuous, and the Theorem of the Maximum implies that the function

$$\inf_{x \in X} \sup_{x' \in X} \phi(x, x', \hat{u}_1, \ldots, \hat{u}_n)$$

is continuous in $(\hat{u}_1, \ldots, \hat{u}_n)$. Since this function is equal to zero at $(u_1, \ldots, u_n)$, by the assumption that the core at $(u_1, \ldots, u_n)$ is nonempty, it follows that it goes to zero as m increases. Furthermore, for each m, the infimum over x is achieved at some x^m. The terms $\Delta(u_1^m, \ldots, u_n^m)$ converge to $\Delta(u_1, \ldots, u_n)$, and it follows that for high enough m, x^m satisfies condition (i) of Corollary 1. By compactness, the sequence $\{x^m\}$ has a convergent subsequence, which, by the Theorem of the Maximum, must have limit x^*. Let x^0 be any policy distinct from x^*, so x^* is majority preferred to x^0 at $(u_1, \ldots, u_n)$. Then we may choose m high enough so that x^m satisfies (i) as well as (ii) of Theorem 2, delivering the result.

3.2 Non-existence of Equilibria

In contrast to Theorem 1, which establishes an abundance of subgame perfect equilibria when voter discount factors exceed one half, the next result shows that if voter discount factors lie strictly below a critical level, then subgame perfect equilibria satisfying our refinements do not exist. (Note that we do not impose party symmetry in the theorem.) We know from Theorem 2 that when the subcore is nonempty and voters have positive, though possibly small, discount factors, equilibria do exist. Indeed, when the core is nonempty, our next result has no bite, as the critical discount factor is zero. As long as the core is empty, however, it tells us that the equilibrium existence problem for one-shot elections carries over to the dynamic setting if voters are sufficiently impatient.

Theorem 3 *Assume for every policy x there exists a policy x' and a majority coalition C such that for all voters $i \in C$,*

$$\frac{u_i(x') - u_i(x)}{\overline{u}_i - \underline{u}_i + u_i(x') - u_i(x)} \quad > \quad \delta_i.$$

Then there does not exist a subgame perfect equilibrium satisfying (OS) and (WD).

It is interesting to compare Theorem 3 to Theorem 1. Since the latter result gives conditions under which equilibria exist, the sufficient condition there must be inconsistent with the condition of Theorem 3. Indeed, the inequality there implies that some voters' discount factors are less than one half. Since Theorem 3 allows for patient parties, the same relationship must hold between Theorems 2 and 3. Let x^* be as in Theorem 2. Then the condition of Theorem 3 implies the existence of x' and a majority C such that for all $i \in C$,

$$\frac{u_i(x') - u_i(x^*)}{\overline{u}_i - \underline{u}_i + u_i(x') - u_i(x^*)} \quad > \quad \delta_i.$$

But then the condition of Theorem 2 implies the existence of majority C' and policies r and s such that for all $i \in C'$,

$$\frac{u_i(x') - u_i(x^*)}{u_i(r) - u_i(s) + u_i(x') - u_i(x^*)} \quad < \quad \delta_i.$$

Since C and C' are both majorities, there is some voter i common to both, but $\overline{u}_i - \underline{u}_i \geq u_i(r) - u_i(s)$, a contradiction. Thus, the conditions of the theorems are indeed incompatible.

We can use Theorem 3 to express the critical discount factor level for voters in terms of a measure of how close the core is to being nonempty, but now our measure is slightly different than before. Define

$$\psi(x) \quad = \quad \sup_{x' \in X} \max_{C \in \mathcal{M}} \min_{i \in C} u_i(x') - u_i(x).$$

In contrast to our definition of ψ°, we now maximize over majority coalitions and minimize over members of the coalition. Note that ψ is non-negative and that $\psi(x) = 0$ if and only if $x \in K$. Intuitively, ψ measures how far the policy x is from being in the core, given the utility functions $u_1, \ldots, u_n$. Accordingly, $\inf_{x\in X} \psi(x)$ measures how far voter utilities are from admitting a nonempty core. To see the relationship between the two measures, fix x' and index voters $i(1), \ldots, i(n)$ in increasing order of the quantity $u_i(x') - u_i(x)$, i.e., $u_{i(k)}(x') - u_{i(k)}(x) \leq u_{i(k+1)}(x') - u_{i(k+1)}(x)$ for all $k = 1, \ldots, n-1$. Then it is straightforward to show that

$$\min_{C\in\mathcal{M}} \max_{i\in C} u_i(x') - u_i(x) \quad = \quad u_{i(\lceil \frac{n+1}{2} \rceil)}(x') - u_{i(\lceil \frac{n+1}{2} \rceil)}(x)$$

and

$$\max_{C\in\mathcal{M}} \min_{i\in C} u_i(x') - u_i(x) \quad = \quad u_{i(\lfloor \frac{n+1}{2} \rfloor)}(x') - u_{i(\lfloor \frac{n+1}{2} \rfloor)}(x).$$

Therefore, we have $\psi^\circ(x) \geq \psi(x)$ for all x, with the two measures being equal when the number of voters is odd.

Because there is no issue about whether a certain policy is majority preferred to another, we can now write our sufficient conditions in a more compact form than in Corollary 1.

Corollary 4 *Assume that for all voters we have*

$$\frac{\inf_{x\in X} \psi(x)}{\overline{u}_i - \underline{u}_i + \inf_{x\in X} \psi(x)} \quad > \quad \delta_i.$$

Then there does not exist a subgame perfect equilibrium satisfying (OS) and (WD).

Note that the critical discount factor level in Corollary 4 is decreasing in our measure of distance from nonemptiness of the core, and that it in fact goes to zero with that measure. Thus, it becomes more difficult to obtain non-existence when the core is closer to nonempty. On the other hand, if we fix voter utilities $(u_1, \ldots, u_n)$ so that the core is empty, then the next corollary confirms that equilibria fail to exist when voter discount factors fall below a given positive level. As a consequence, only those policy paths in the core are supportable for arbitrarily low voter discount factors.

Corollary 5 *Assume that X is a compact topological space, that u_i is continuous for all voters, and that $K = \emptyset$. Then there exists $\underline{\delta} > 0$ such that, when $\delta_i < \underline{\delta}$ for all voters, there does not exist a subgame perfect equilibrium satisfying (OS) and (WD).*

To prove the corollary, we invoke the Theorem of the Maximum to obtain continuity of ψ. Since X is compact and $\inf_{x\in X} \psi(x) > 0$, it follows that ψ achieves its minimum at some policy $\underline{x}$. Since this minimum is positive, the critical level in Corollary 4 is positive, so we may set

$$\underline{\delta} \quad = \quad \frac{\psi(\underline{x})}{\max_{i\in N}(\overline{u}_i - \underline{u}_i) + \psi(\underline{x})}$$

to fulfill the claim of the corollary.

We can refine Theorem 3 if we add the assumption that parties are relatively impatient. Assuming party discount factors are less than one half, we can strengthen the conclusion of Theorem 3 in two ways: we can state the result for particular policy paths, and we can state the result using the existential quantifier over t (rather than the universal quantifier over x). The latter improvement is possible because, when parties are impatient, it is sufficient that either party can deviate and win in at least one period.

Theorem 4 *Let $\delta_P < 1/2$, and let $\mathbf{x}$ be a policy path. Assume there exists a period t, a policy x', and a majority coalition C such that for all voters $i \in C$,*

$$\frac{u_i(x') - u_i(x_t)}{\overline{u}_i - \underline{u}_i + u_i(x') - u_i(x_t)} \quad > \quad \delta_i.$$

Then the path $\mathbf{x}$ is not supportable by a subgame perfect equilibrium satisfying (OS) and (WD).

As above, we can formulate Theorem 4 in terms of a measure of distance from the core. Now, however, our measure takes the supremum over time periods, rather than an infimum over policies, reflecting the fact that the proof of the theorem may operate on the policy on the path furthest from being in the core. In other words, we are measuring how far the entire *path* is from lying in the core.

Corollary 6 *Let $\delta_P < 1/2$ for the parties, and let $\mathbf{x}$ be a policy path. Assume that for all voters we have*

$$\frac{\sup_{t\in\mathbb{N}} \psi(x_t)}{\overline{u}_i - \underline{u}_i + \sup_{t\in\mathbb{N}} \psi(x_t)} \quad > \quad \delta_i.$$

Then the path $\mathbf{x}$ is not supportable by a subgame perfect equilibrium satisfying (OS) and (WD).

As before, the critical discount factor level in Corollary 6 is decreasing in our measure of distance from the core, and it in fact goes to zero with that measure. Thus, it becomes more difficult to support paths that are further from lying in the core. If the core is empty, then we have $\sup_{t\in\mathbb{N}} \psi(x_t) > 0$ for all paths, and by a Theorem of the Maximum argument, we can show that equilibria fail to exist when voter discount factors fall below a given positive level. We omit the formal statement of this result, as it follows the lines above.

3.3 Core Equivalence

We have seen from Proposition 3 that strong stationary pins down subgame perfect equilibria substantially, if they exist: a policy path is supportable if and only if it lies in the plurality core. In this section, we drop strong stationarity and seek weaker conditions under which the equilibrium outcomes of our model are equivalent to the core. An implication of

Proposition 3 is that every policy path through the plurality core can be supported by a subgame perfect equilibrium, in fact, by a strongly stationary one. In the remainder of this section, we give sufficient conditions for the opposite inclusion, namely, that all supportable policy paths must lie in the core. An implication of Corollary 5 is that only paths in the core can be supported for arbitrarily small voter discount factors, but we are interested in core equivalence for a non-trivial range of discount factors. Clearly, Theorem 1 implies that voters cannot have discount factors greater than one half. And since we consider an environment in which the core is nonempty, Theorem 2 implies that party discount factors also must be less than one half.

Our final result imposes these restrictions on discount factors, with a slight strengthening in the case of party discount factors: we assume they are less than one third. We add the following strong, but standard, conditions: the number of voters is odd, the policy space is Euclidean, voter utilities are quadratic, i.e., $u_i(x) = -||x - \hat{x}_i||^2$ for each i, where $\hat{x}_i$ is i's "ideal point," and the ideal points of the voters are distinct, i.e., $\hat{x}_i = \hat{x}_j$ implies $i = j$. Furthermore, the theorem requires the non-emptiness of the core, an assumption that is automatically satisfied in one dimension but quite restrictive in multiple dimensions. Recall that under these conditions, the core consists of a single point, the ideal point of the "core" voter.

Theorem 5 *Assume that n is odd, that the utility functions u_i are quadratic with distinct ideal points, and that $K \neq \emptyset$. Let $\delta_i = \delta < 1/2$ for all voters, and let $\delta_P < 1/3$ for the parties. Then $\mathbf{x}$ is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS) if and only if $x_t \in K$ for all t.*

Thus, for a range of discount factors, we essentially obtain uniqueness of subgame perfect equilibria satisfying our refinements, in contrast to the stark multiplicity results of Theorems 1 and 2. The parties must locate at the core point, lending a non-cooperative foundation to the social choice concept of the core and, for the one-dimensional special case, providing a game-theoretic version of the median voter theorem in the context of repeated Downsian elections. The proof uses a result of Banks and Duggan (2006b) on the decisiveness of the core voter and focuses on necessary conditions satisfied by equilibrium policies furthest from the core. Because we assume party discount factors are less than one third, rather than one half, Theorem 5 leaves a slight gap. We conjecture that the core equivalence result extends to this region of the parameter space, but we leave the question open.

4 Conclusion

We have shown that, if voters are relatively patient, or if parties are relatively patient and the subcore is close to non-empty, then there is a subgame perfect equilibrium of the infinitely repeated electoral game. This is true regardless of the dimensionality of the policy space or voter preferences, providing a solution to the equilibrium existence problem. This sword is double-edged, however, for, in fact, *every* path of policies can be supported by a refinement of subgame perfect equilibria. As a consequence, the sharp predictions of the median voter theorem — and more generally core equivalence in multiple dimensions with a non-empty core — are endangered. We show that the median voter theorem holds if parties and voters are sufficiently impatient, but then we lose existence of equilibria in multiple dimensions when the core is empty. To achieve a general equilibrium existence result that preserves the Median Voter Theorem in a model of infinitely repeated elections, we conclude that the background assumptions of the Downsian model must be re-examined. As in the electoral accountability approach, alternatives may involve policy motivations for candidates, dropping the commitment assumption (as in the literature on citizen-candidates), allowing for imperfect information about voter preferences (as in the literature on probabilistic voting), or some combination of these directions.

A Proofs of Theorems

Theorem 1 *Let $\delta_i > 1/2$ for all voters. Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

Proof: Let $\mathbf{x} = (x_1, x_2, \ldots)$ be any policy path. We construct a subgame perfect equilibrium to support $\mathbf{x}$ by labeling each complete history and each partial history. The labeling rule, defined recursively below, will simplify our specification of strategies. We begin by labeling the initial (complete) history with P-Eq(0). Next, we label all partial histories. If a complete history h_{t-1} is labeled P-Eq($t-1$), then we interpret this to mean "the parties have followed the desired path of play through period $t-1$ and will continue to do so." In this case, given platforms (y, z), label the partial history (h_{t-1}, y, z) as follows:

- ADev(y, x_t) if $y\ M^*\ x_t = z$,
- BDev(z, x_t) if $z\ M^*\ x_t = y$, and
- V-Eq(t) in all other cases.

Thus, if the parties both choose x_t, or if one party deviates to something not plurality-preferred to x_t, or if both parties deviate to other platforms, then the labeling continues to reflect that we follow the desired path of play. On the other hand, if a party, say A, deviates to some y such that $y\ M^*\ x_t$, then we label the partial history with ADev(y, x_t).

If a complete history h_{t-1} is labeled P-Absorb(x), then we interpret this to mean "the parties are supposed to choose x in period t and will do so ever after." Given platforms (y, z), label the partial history (h_{t-1}, y, z) as follows:

- ADev(y, x) if $y\ M^*\ x = z$,
- BDev(z, x) if $z\ M^*\ x = y$, and
- V-Absorb(x) in all other cases.

Once again, the labeling changes if a party, say A, deviates to some y that is plurality-preferred to x.

Lastly, we label all complete histories (other than the initial history). If a partial history (h_{t-1}, y, z) is labeled V-Eq(t), then we interpret this to mean that "the parties chose the specified equilibrium policies or that no party has unilaterally deviated to a plurality preferred policy." Here, the labeling of the complete history does not depend on who wins. That is, the resulting complete history h_t is labeled P-Eq(t).

Similarly, if a partial history (h_{t-1}, y, z) is labeled V-Absorb(x), then the resulting complete history h_t is labeled P-Absorb(x). We interpret the label VAbsorb(x) to mean "the parties chose policy x, as required, or no party has unilaterally deviated to a plurality preferred policy." In this case, the labeling of the resulting complete history is independent of the outcome of voting.

If the partial history (h_{t-1}, y, z) is labeled ADev(y, x), then we interpret this to mean "the parties were both supposed to choose x but A deviated to y." In this case, the labeling of the resulting complete history depends on which party wins the election in period t. We label the complete history h_t as follows:

- P-Absorb(x) if A wins the election in period t, and
- P-Absorb(y) if B in period t.

Thus, if A deviates from x to y and wins, then the resulting history is labeled P-Absorb(x), i.e., the original policy outcome ever after. If A deviates and B wins, then the resulting history is labeled P-Absorb(y), i.e., A's deviation forever. Since we suppose $y \ M^* \ x$, this will give a plurality of voters an incentive to vote against A.

Similarly, if a partial history (h_{t-1}, y, z) is labeled BDev(z, x), then label the complete history h_t as follows:

- P-Absorb(x) if B wins the election in period t, and
- P-Absorb(z) if A in period t.

We next specify strategies for parties and voters.

1. Parties:
   (a) If h_{t-1} is labeled P-Eq$(t-1)$, then the parties adopt platforms $y_t = z_t = x_t$.
   (b) If h_{t-1} is labeled P-Absorb(x), then both adopt $y_t = z_t = x$.
2. Voters:
   (a) If the partial history (h_{t-1}, y, z) is labeled V-Eq(t) or V-Absorb(x), then voter i votes for A if $u_i(y) > u_i(z)$; i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(y) = u_i(z)$.
   (b) If (h_{t-1}, y, z) is labeled ADev(y, x), then voter i votes for A if $u_i(x) > u_i(y)$; i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(x) = u_i(y)$.

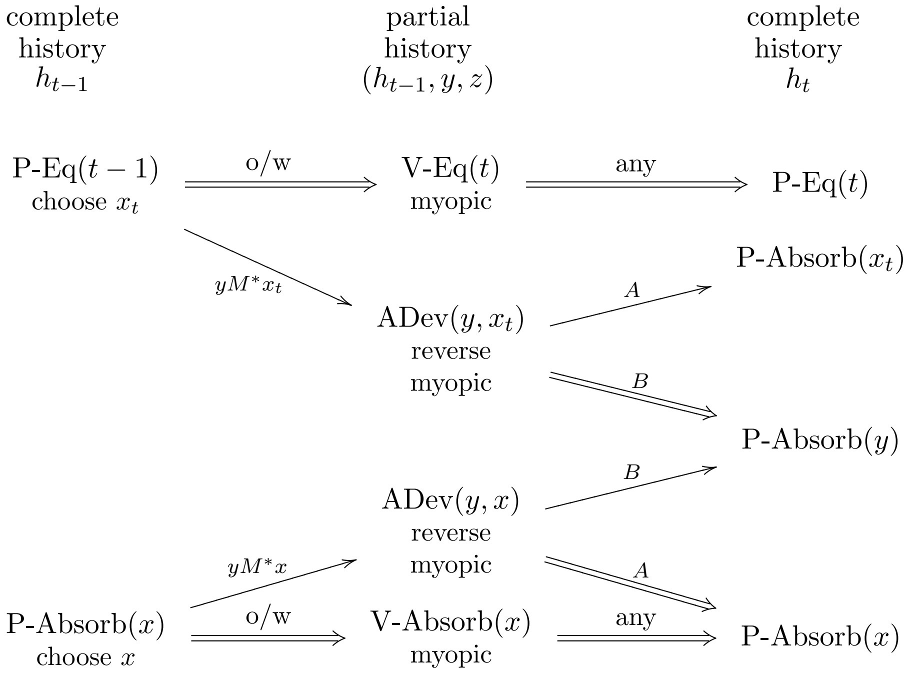

Figure 1: Labels and strategies in the proof of Theorem 1

(c) If (h_{t-1}, y, z) is labeled BDev(z, x), then voter i votes for B if $u_i(x) > u_i(z)$; i votes for A if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(x) = u_i(z)$.

The labeling rule and strategies we use are diagrammed in Figure 1, where arrows denote the labeling rule for partial and complete histories as a function of party platforms and election returns. The text below the labels give the equilibrium actions for parties and voters. Double arrows indicate equilibrium strategies. Here, "myopic" indicates that voters vote for the policy they prefer in the current period, while "reverse myopic" indicates that voters vote against the policy they prefer in the current period. For clarity, we only include deviations by A in the figure.

We now verify that the above specification of strategies is, indeed, subgame perfect and satisfies (OS), (WD), and (PS). To establish subgame perfection, by the one-shot deviation principle (Fudenberg and Tirole, 1991), we need show only that no party or voter can achieve a higher expected discounted payoff by a "one-shot deviation" following any history. That is, we need to show, given an arbitrary history, that no party or voter can profit by deviating in the following period and returning to the above strategy thereafter. Consider a voter i's decision after a partial history (h_{t-1}, y, z) labeled V-Eq(t). Regardless of the winner in period t, according to the above strategies, the parties will choose the same platforms in subsequent periods, namely, $x_{t+1},\ x_{t+2},\ \ldots,$ so $v_i(h_{t-1}, y, z, 1) = v_i(h_{t-1}, y, z, 0)$. It is a best response to vote for A if

$$(1-\delta_i)u_i(y) + \delta_i v_i(h_{t-1}, y, z, 1) \quad \geq \quad (1-\delta_i)u_i(z) + \delta_i v_i(h_{t-1}, y, z, 0),$$

which is equivalent to $u_i(y) \geq u_i(z)$, and voting for B is a best response if $u_i(z) \geq u_i(y)$. Similarly, after a partial history (h_{t-1}, y, z) labeled V-Absorb(x), the above strategies specify that the parties both choose x forever, and thus 2(a) is a best response.

After a partial history (h_{t-1}, y, z) labeled ADev(y, x), the policy path depends on which party wins in period t. If A wins, then, according to the above strategies, both parties will choose x thereafter, so $v_i(h_{t-1}, y, z, 1) = u_i(x)$; if B wins, then both parties will choose y thereafter, so $v_i(h_{t-1}, y, z, 1) = u_i(y)$. Thus, it is a best response for i to vote for A if

$$(1-\delta_i)u_i(y) + \delta_i u_i(x) \quad \geq \quad (1-\delta_i)u_i(x) + \delta_i u_i(y).$$

Since $\delta_i > 1/2$, this is equivalent to $u_i(x) \geq u_i(y)$, and it is a best response to vote for A if $u_i(x) \geq u_i(y)$, likewise for B if $u_i(y) \geq u_i(x)$, as in 2(b). Note that, since $y \; M^* \; x$ by construction, the latter holds for a plurality of voters, so B will win with probability greater than one half in period t. The analysis is similar after a partial history (h_{t-1}, y, z) labeled BDev(z, x), but, in these cases, A wins with probability greater than one half in period t. We conclude that the strategies specified above for voters are best responses after all histories.

We now turn to the parties. Consider the decision of a party, say A, after a history h_{t-1} labeled P-Eq$(t-1)$. According to the strategies specified above, the parties both choose x_t in period t and follow $\mathbf{x}$ thereafter, the voters flip coins to decide between parties in all periods, and A's expected discounted payoff is one half. If A deviates by choosing platform $y \neq x_t$ and following the above strategy thereafter, then there are two possibilities. First, if $y \; M^* \; x_t$, then (h_{t-1}, y, x_t) is labeled ADev(y, x_t). By 2(b), with some probability $\pi_A < 1/2$, party A wins, the resulting complete history h_t is labeled P-Absorb(x_t), and the voters randomize thereafter, giving the party an expected discounted payoff of one half. With probability $\pi_B = 1 - \pi_A > 1/2$, party B wins, and the new history is labeled P-Absorb(y). After that history, according to 1(b) above, both parties choose y and, by 2(a), voters randomize between the parties thereafter. Thus, A's expected discounted payoff from deviating is

$$\begin{aligned} &\pi_A((1-\delta_A)(1) + \delta_A(1/2)) + \pi_B((1-\delta_A)(0) + \delta_A(1/2) \\ &= \quad (1-\delta_A)(\pi_A) + \delta_A(1/2), \end{aligned}$$

which is less than one half, since $\pi_A < 1/2$. Second, if not $y \; M^* \; x_t$, then (h_{t-1}, y, x_t) is labeled with P-Eq(t). By 2(a), A wins with probability $\pi_A \leq 1/2$ and B wins with probability $\pi_B \geq 1/2$. By 1(a) and 2(a), both parties follow the path $\mathbf{x}$ and voters randomize between them thereafter. Thus, A's expected discounted payoff from deviating is less than or equal to one half. The logic following a history h_t labeled P-Absorb(x) is similar: if a party deviates to a platform plurality-preferred to x, then it will win in period t with probability less than one half and win half the time thereafter; if it deviates to a platform not plurality-preferred to x, then it can do no better than win half the time in t, and it wins half the time thereafter. We conclude that party A, likewise party B, has no profitable one-shot deviations.

Thus, the above specification of strategies is a subgame perfect equilibrium, and it clearly supports $\mathbf{x}$. That it satisfies (OS) follows from two observations: the labeling rule for complete histories only depends on past electoral outcomes; and strategies, in turn, only depend on labels. Finally, (WD) and (PS) are clear from the preceding arguments. ∎

Theorem 2 *Let $\delta_P \geq 1/2$ for both parties. Assume there exists a policy x^* such that (i) $x^*\ M\ x^0$ for some policy x^0, and (ii) for every policy x', there exist a majority coalition C and policies r and s such that for all voters $i \in C$, we have $u_i(r) \geq u_i(s)$ and, in case $u_i(x') \geq u_i(x^*)$, we also have*

$$\frac{u_i(x') - u_i(x^*)}{u_i(r) - u_i(s) + u_i(x') - u_i(x^*)} \quad < \quad \delta_i.$$

Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).

Proof: Let x^* be as in the statement of the theorem, let x^0 be such that $x^*\ M\ x^0$, and for every x', let $C(x')$, $r(x')$, and $s(x')$ fulfill the condition of the theorem. Let $\mathbf{x} = (x_1, x_2, \ldots)$ be any policy path. As in the proof of Theorem 1, we label each complete history and each partial history. Following the proof of Theorem 1, the labeling rule is defined recursively as follows. The initial (complete) history is labeled P-Eq(0). As before, a complete history h_{t-1} labeled P-Eq($t-1$) is interpreted to mean "the parties have followed the desired path of play through period $t-1$ and will continue to do so." In this case, label the partial history (h_{t-1}, y, z) as follows:

- ADev if $y\ M^*\ x_t = z$,
- BDev if $z\ M^*\ x_t = y$, and
- V-Eq(t) in all other cases.

Similarly, if a complete history h_{t-1} is labeled P-Absorb(x^*), then we interpret this to mean "the parties are supposed to choose x^* in period t and will do so ever after." In this case, label the partial history (h_{t-1}, y, z) as follows:

- ADev if $y\ M^*\ x^* = z$,
- BDev if $z\ M^*\ x^* = y$, and
- V-Absorb(x^*) in all other cases.

If a complete history h_{t-1} is labeled P-ADevWin, then we interpret this to mean "party A deviated from the desired path in the previous period and won the election," and similarly for P-BDevWin. If the complete history h_{t-1} is labeled P-ADevWin, label the partial history (h_{t-1}, y, z) as follows:

- ADevDev(y) if $y \neq z = x^*$,

- BDevDev(z) if $z \neq y = x^*$, and
- V-ADevWin in all other cases.

Similarly, if the complete history h_{t-1} is labeled P-BDevWin, then label the partial history (h_{t-1}, y, z) as follows:

- ADevDev(y) if $y \neq z = x^*$,
- BDevDev(z) if $z \neq y = x^*$, and
- V-BDevWin in all other cases.

If a complete history h_{t-1} is labeled P-DDev(x), then we interpret this to mean "some party deviated from the desired path of play; this was followed by another deviation; and from now on the parties will both choose x." In this case, label the partial history (h_{t-1}, y, z) as follows:

- ADev if $y\ M^*\ z = x$,
- BDev if $z\ M^*\ y = x$, and
- V-DDev(x) in all other cases.

Lastly, we label all complete histories (except the initial history). We denote by P the winning party following h_{t-1} and platform choices (y, z). If a partial history (h_{t-1}, y, z) is labeled V-Eq(t), then we interpret this to mean "the parties chose the specified equilibrium policies or no party has unilaterally deviated to a plurality preferred policy." Here, the labeling of the complete history does not depend on who wins. That is, the resulting complete history h_t is labeled P-Eq(t).

Similarly, if a partial history (h_{t-1}, y, z) is labeled V-Absorb(x), then the resulting complete history h_t is labeled P-Absorb(x). We interpret the label VAbsorb(x) to mean "the parties chose policy x, as required, or no party has unilaterally deviated to a plurality preferred policy." In this case, the labeling of the resulting complete history is independent of the outcome of voting.

If the partial history (h_{t-1}, y, z) is labeled ADev, then we interpret this to mean "party A deviated from the desired path of play in the previous period." In this case, the labeling of the resulting complete history depends on which party wins the election in period t. We label the complete history h_t as follows:

- P-ADevWin if $P = A$, and
- P-Absorb(x^*) if $P = B$.

Likewise, if the partial history (h_{t-1}, y, z) is labeled BDev, we label the complete history h_t as follows:

- P-Absorb(x^*) if $P = A$, and
- P-ADevWin if $P = B$.

If the partial history (h_{t-1}, y, z) is labeled V-ADevWin, then we interpret this to mean "party A deviated from the desired path in a previous period and won." We label the complete history h_t as follows:

- P-DDev(x^0) if $P = A$, and
- P-ADevWin if $P = B$.

Likewise, if the partial history (h_{t-1}, y, z) is labeled V-BDevWin, then we label the complete history h_t as follows:

- P-BDevWin if $P = A$, and
- P-DDev(x^0) if $P = B$.

If the partial history (h_{t-1}, y, z) is labeled ADevDev(y), then we interpret this to mean "after a previous deviation from the desired path, and party A has deviated in period t to y." In this case, the labeling of the resulting complete history again depends on the winning party in period t. We label the complete history h_t as follows:

- P-DDev($s(y)$) if $P = A$, and
- P-DDev($r(y)$) if $P = B$.

Likewise, if the partial history (h_{t-1}, y, z) is labeled BDevDev(z), then we label the complete history h_t as follows:

- P-DDev($r(y)$) if $P = A$, and
- P-DDev($s(y)$) if $P = B$.

Finally, if the partial history (h_{t-1}, y, z) is labeled V-DDev(x), then we interpret this to mean "some party deviated from the desired path of play, followed by another deviation, and the parties are to choose x forever." In this case, the labeling of the complete history does not depend on who wins. That is, the resulting complete history h_t is labeled P-DDev(x).

We next specify strategies for parties and voters.

1. Parties:
   (a) If h_{t-1} is labeled P-Eq($t-1$), then the parties adopt platforms $y_t = z_t = x_t$.
   (b) If h_{t-1} is labeled P-Absorb(x^*) or P-ADevWin or P-BDevWin, then both adopt $y_t = z_t = x^*$.

(c) If h_{t-1} is labeled P-DDev(x), then both adopt $y_t = z_t = x$.

2. Voters:

(a) If the partial history (h_{t-1}, y, z) is labeled V-Eq(t), V-Absorb(x^*), ADev, BDev, or V-DDev(x), then voter i votes for A if $u_i(y) > u_i(z)$; i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(y) = u_i(z)$.

(b) If (h_{t-1}, y, z) is labeled V-ADevWin, then voter i votes for B if

$$(1-\delta_i)u_i(z) + \delta_i u_i(x^*) > (1-\delta_i)u_i(y) + \delta_i u_i(x^0);$$

voter i votes for A if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

(c) If (h_{t-1}, y, z) is labeled V-BDevWin, then voter i votes for A if

$$(1-\delta_i)u_i(y) + \delta_i u_i(x^*) > (1-\delta_i)u_i(z) + \delta_i u_i(x^0);$$

voter i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

(d) If (h_{t-1}, y, z) is labeled ADevDev(y), then voter i votes for B if

$$(1-\delta_i)u_i(z) + \delta_i u_i(r(y)) > (1-\delta_i)u_i(y) + \delta_i u_i(s(y));$$

voter i votes for A if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

(e) If (h_{t-1}, y, z) is labeled BDevDev(z), then voter i votes for A if

$$(1-\delta_i)u_i(y) + \delta_i u_i(r(z)) > (1-\delta_i)u_i(z) + \delta_i u_i(s(y));$$

voter i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

As in the previous proof, we illustrate the labeling rule and strategies in Figure 2. As before, arrows denote the labeling rule for partial and complete histories as a function of party platforms and election returns. The text below the labels give the equilibrium actions for parties and voters. Double arrows indicate equilibrium strategies. Here, "myopic" indicates that voters vote for the policy they prefer in the current period, while "2(b)" and "2(d)" refer to the strategies listed above in which voters anticipate the future choices of parties. For clarity, we only include deviations by A in the figure.

To show that these strategies form a subgame perfect equilibrium, it is sufficient, by the one-shot deviation principle, to show there is no history at which a party or voter can profitably deviate that period and return to the above strategy thereafter. Consider a voter i's decision after a partial history (h_{t-1}, y, z) labeled V-Eq(t). Regardless of the winner in period t, according to the above strategies, the parties will choose the same platforms thereafter, namely, $x_{t+1}, x_{t+2}, \ldots$, so $v_i(h_{t-1}, y, z, 1) = v_i(h_{t-1}, y, z, 0)$. It is a best response to vote for A if

$$(1-\delta_i)u_i(y) + \delta_i v_i(h_{t-1}, y, z, 1) \quad \geq \quad (1-\delta_i)u_i(z) + \delta_i v_i(h_{t-1}, y, z, 0),$$

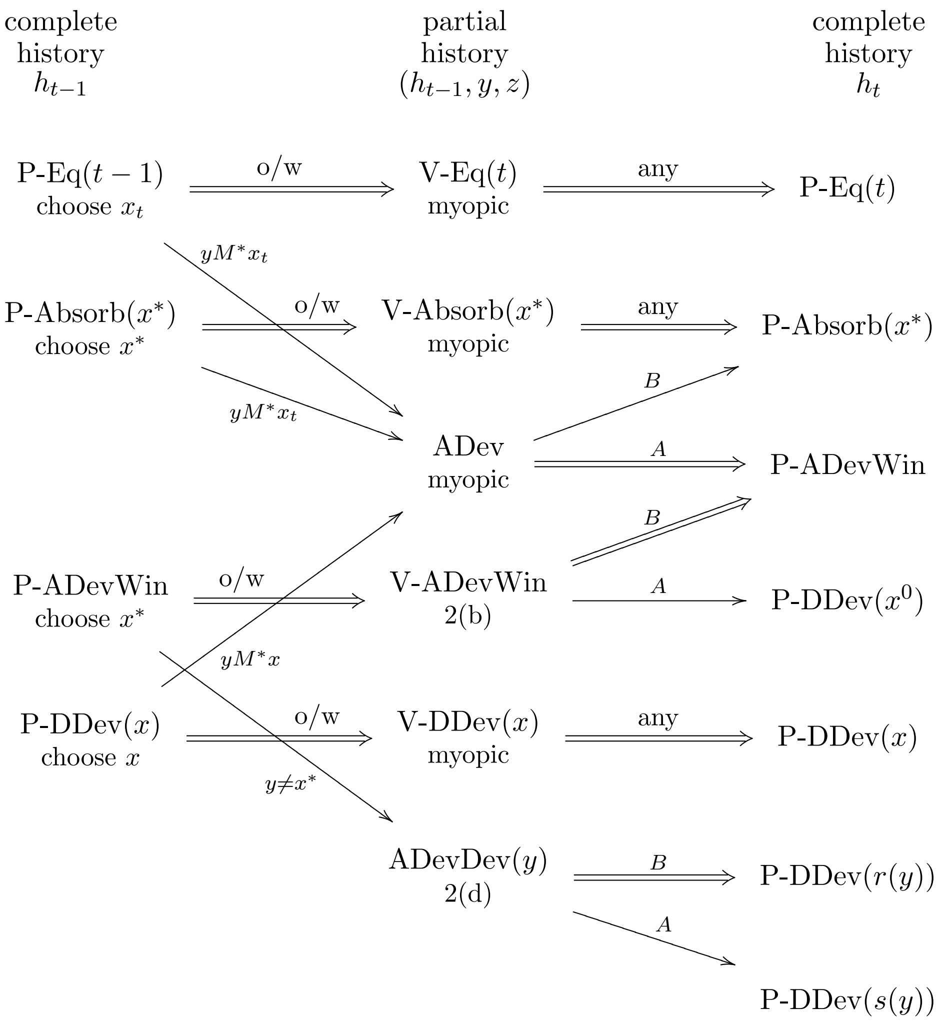

Figure 2: Labels and strategies in the proof of Theorem 2

which is equivalent to $u_i(y) \geq u_i(z)$, and voting for B is a best response if $u_i(z) \geq u_i(y)$, so 2(a) gives the voters' best responses. Similarly, after (h_{t-1}, y, z) labeled V-Absorb(x^*), the above strategies specify that the parties both choose x^* forever, and 2(a) is a best response. After (h_{t-1}, y, z) labeled V-DDev(x), the parties both choose x forever, so 2(a) is a best response. Next, consider a partial history (h_{t-1}, y, z) labeled ADev. If B wins, then the complete history h_t is labeled P-Absorb(x^*), and both parties choose x^* thereafter. Thus, $v_i(h_{t-1}, y, z, 0) = u_i(x^*)$. If A wins, then h_t is labeled P-ADevWin, and, by 1(b), the parties both choose x^* in period $t + 1$. Then, by 2(b), voter i votes for B if $u_i(x^*) > u_i(x^0)$, so a majority of voters vote for B, and B wins with probability one. According to our labeling rule, future histories are labeled V-ADevWin and P-ADevWin, and B continues to win thereafter with platform x^*. Thus, $v_i(h_{t-1}, y, z, 1) = u_i(x^*)$. Again, voting for A is a best response if $u_i(y) \geq u_i(z)$, and voting for B is a best response if $u_i(z) \geq u_i(y)$, so 2(a) is a best response. Applying the same argument, 2(a) is also a best response for a partial history labeled BDev.

After (h_{t-1}, y, z) labeled V-ADevWin, the policy path depends on which party wins in period t. If B wins, then the complete history h_t is labeled P-ADevWin, and the above strategies specify that both parties will choose x^* thereafter, so $v_i(h_{t-1}, y, z, 0) = u_i(x^*)$. If A wins, then h_t is labeled P-DDev(x^0), where, by 1(c), the parties both choose x^0 thereafter, implying $v_i(h_{t-1}, y, z, 1) = u_i(x^0)$. Thus, it is a best response for i to vote for B if

$$(1 - \delta_i)u_i(z) + \delta_i u_i(x^*) \quad \geq \quad (1 - \delta_i)u_i(y) + \delta_i u_i(x^0),$$

and 2(b) is a best response. For V-BDevWin, the same argument shows that 2(c) is a best response.

Now consider a partial history (h_{t-1}, y, z) labeled ADevDev(y). If B wins in period t, the complete history h_t is labeled P-DDev($r(y)$ and both parties adopt $r(y)$ in all following periods. This implies that $v_i(h_{t-1}, y, z, 0) = u_i(r(y))$. If A wins, then h_t is labeled P-DDev($s(y)$, where both parties choose $s(y)$ thereafter, so $v_i(h_{t-1}, y, z, 1) = u_i(s(y))$. Thus, it is a best response for i to vote for B if

$$(1 - \delta_i)u_i(z) + \delta_i u_i(r(y)) \quad \geq \quad (1 - \delta_i)u_i(y) + \delta_i u_i(s(y)),$$

and voting for A is a best response if the opposite weak inequality holds, as in 2(d). Note that if $x^* = z$, then for all $i \in C(y)$ with $u_i(y) \geq u_i(x^*)$, we have

$$\frac{u_i(y) - u_i(x^*)}{u_i(r(y)) - u_i(s(y)) + u_i(y) - u_i(x^*)} \quad < \quad \delta_i$$

by assumption. Using $u_i(r(y)) \geq u_i(s(y))$, this implies that for all $i \in C(y)$, we have

$$(1 - \delta_i)u_i(x^*) + \delta_i u_i(r(y)) \quad > \quad (1 - \delta_i)u_i(y) + \delta_i u_i(s(y)).$$

Therefore, by 2(d), a majority of voters vote for B, and B wins with probability one. A similar calculation shows that 2(e) is a best response at a partial history labeled BDevDev(z). In that case, if $y = x^*$, then a majority of voters vote for A, and A wins with probability one. We conclude that the above voting strategies specify best responses after all histories.

Turning to the parties, consider the decision of a party, say A, after a complete history h_{t-1} labeled P-Eq($t - 1$). According to the strategies specified above, the parties both choose x_t in period t and follow $\mathbf{x}$ thereafter, the voters flip coins to decide between parties in all periods, and A's expected discounted payoff is one half. If A deviates to y such that $y\ M^*\ x_t$, then the partial history (h_{t-1}, y, z) is labeled ADev and, by 2(a), party A wins with probability $\pi_A > 1/2$ in period t. The resulting complete history h_t is then labeled P-ADevWin, and, according to the strategies specified above, A wins with probability zero thereafter. Party B wins with probability $\pi_B = 1 - \pi_A < 1/2$ in period t, after which the history h_t is labeled P-Absorb(x^*), and A's expected discounted payoff is then one half. Thus, A's expected discounted payoff from deviating is

$$\pi_A((1-\delta_A)(1) + \delta_A(0)) + \pi_B(1/2),$$

which is no greater than one half, since $\delta_A \geq 1/2$. If A deviates to some y that is not plurality preferred to x_t, then the resulting partial history is labeled V-Eq(t). By 2(a), A wins in period t with probability less than or equal to one half. Regardless of the winner, h_t is labeled P-Eq(t), and A's expected discounted payoff is then one half. Thus, the deviation is not profitable.

After a complete history labeled P-Absorb(x^*), according to the above strategies, both parties choose x^* and A wins with probability one half in every period, yielding an expected discounted payoff of one half. If A deviates to a policy y such that $y\ M^*\ x^*$, then the resulting partial history is labeled ADev, and the argument proceeds as following a deviation from a history labeled P-Eq($t-1$). If A deviates to some y that is not plurality preferred to x_t, then the resulting partial history is labeled V-Absorb(x^*). By 2(a), A wins in period t with probability less than or equal to one half. Regardless of the winner, h_t is labeled P-Absorb(x^*), and A's expected discounted payoff is again one half, so the deviation is not profitable.

Next, take a complete history h_{t-1} labeled V-ADevWin. According to the strategies specified above, party A wins with probability zero in period t and in all future periods. If A deviates to $y \neq x^*$, then the resulting partial history is labeled ADevDev(y), where B chooses x^* and wins with probability one in all future periods. Thus, the deviation is not profitable. After a complete history labeled V-BDevWin, A's expected discounted payoff is one, so A clearly has no profitable deviation.

Last, after a complete history h_{t-1} labeled P-DDev(x), according to the strategies specified above, both parties choose x forever, and A's expected discounted payoff is one half. If A deviates to y such that $y\ M^*\ x$, then the resulting partial history is labeled ADev, and the argument proceeds as following a deviation from a history labeled P-Eq($t-1$). Thus, the deviation is not profitable. We conclude that party A, and likewise party B, has no profitable one-shot deviations.

Thus, the above specification of strategies is a subgame perfect equilibrium, and it clearly supports $\mathbf{x}$. That it satisfies (OS) follows from two observations: the transition rule for states only depends on past electoral outcomes; and strategies, in turn, only depend on states. Finally, (WD) and (PS) are clear from the preceding arguments. ∎

Theorem 3 *Assume for every policy x there exists a policy x' and a majority coalition C*

such that for all voters $i \in C$,

$$\frac{u_i(x') - u_i(x)}{\overline{u}_i - \underline{u}_i + u_i(x') - u_i(x)} \quad > \quad \delta_i.$$

Then there does not exist a subgame perfect equilibrium satisfying (OS) and (WD).

Proof: Consider a subgame perfect equilibrium σ satisfying (OS) and (WD), and suppose that for every $x \in X$, there exists $x' \in X$ and majority coalition $C \in \mathcal{M}$ such that for all voters $i \in C$, the inequality in the statement of the theorem is violated. Consider any complete history h_t, and let $x = \sigma_B(h_t)$. By supposition, there exists x' and majority C such that the inequality in the statement of the theorem holds for all $i \in C$. Then for all $i \in C$, we have

$$(1-\delta_i)u_i(x') + \delta_i \underline{u}_i \quad > \quad (1-\delta_i)u_i(x_t) + \delta_i \overline{u}_i.$$

Given partial history (h_t, x', x), (WD) implies that voter $i \in C$ votes for party A if

$$(1-\delta_i)u_i(x') + \delta_i v_i(x', x, 1) \quad > \quad (1-\delta_i)u_i(x) + \delta_i v_i(x', x, 0),$$

which is implied by the preceding inequality. Therefore, if party A locates at $y = x'$ after complete history h_t, the party wins. Since h_t is arbitrary, we conclude that party A's expected discounted payoff in σ is equal to one. By a symmetric argument, party B's payoff is also one, a contradiction. ∎

Theorem 4 *Let $\delta_P < 1/2$, and let $\mathbf{x}$ be a policy path. Assume there exists a period t, a policy x', and a majority coalition C such that for all voters $i \in C$,*

$$\frac{u_i(x') - u_i(x_t)}{\overline{u}_i - \underline{u}_i + u_i(x') - u_i(x_t)} \quad > \quad \delta_i.$$

Then the path $\mathbf{x}$ is not supportable by a subgame perfect equilibrium satisfying (OS) and (WD).

Proof: Consider a policy path $\mathbf{x}$ supported by a subgame perfect equilibrium satisfying (OS) and (WD), and suppose that there exist t, x', and C such that the inequality in the statement of the theorem holds for all $i \in C$. Letting h_{t-1} denote the equilibrium path of play in the first $t-1$ periods, we have, by assumption, that x_t is the policy outcome in period t with probability one. One of the parties, say A, must have an expected discounted payoff starting from the beginning of period t of less than or equal to one half. Thus, because $\delta_A < 1/2$, party B wins in period t with some positive probability, and we conclude that B's platform is x_t. We claim that A can deviate to x' in period t and win with probability one. Indeed, for all $i \in C$, the inequality

$$(1-\delta_i)u_i(x') + \delta_i v_i(x', x_t, 1) \quad > \quad (1-\delta_i)u_i(x_t) + \delta_i v_i(x', x_t, 0)$$

is implied by

$$(1-\delta_i)u_i(x') + \delta_i \underline{u}_i \quad > \quad (1-\delta_i)u_i(x_t) + \delta_i \overline{u}_i,$$

which is equivalent to the inequality in the statement of the theorem. By (WD), therefore, A wins in period t after deviating to x'. The expected discounted payoff from deviating is at least $(1-\delta_A)(1)+\delta_A(0) = 1-\delta_A > 1/2$, so the deviation is profitable, a contradiction. ∎

Theorem 5 *Assume that n is odd, that the utility functions u_i are quadratic with distinct ideal points, and that $K \neq \emptyset$. Let $\delta_i = \delta < 1/2$ for all voters, and let $\delta_P < 1/3$ for the parties. Then $\mathbf{x}$ is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS) if and only if $x_t \in K$ for all t.*

Proof: That $\mathbf{x}$ can be supported if $x_t \in K$ for all t follows from Proposition 3. Consider any subgame perfect equilibrium, and let X^* be the set of possible equilibrium policy outcomes: $x \in X^*$ if and only if there is some complete history after which one party adopts x and wins with positive probability. By our assumptions that n is odd and that voters have quadratic utility functions, the core contains a single policy and this is the ideal point of some voter, indexed k. By $K \neq \emptyset$, by the assumption of common discount factors, and by (WD), Lemma 1 of Banks and Duggan (2006b) then implies that this core voter is "decisive" after all histories. That is, A wins with probability one after partial history (h_t, y, z) if

$$(1-\delta)u_k(y) + \delta v_k(h_t, y, z, 1) \quad > \quad (1-\delta)u_k(z) + \delta v_k(h_t, y, z, 0),$$

and B wins with probability one if the inequality is reversed. Furthermore, if equality holds for the core voter, then, by the assumption of distinct ideal points, the above inequality and the reverse inequality hold for equal numbers of voters. Thus, in this case, the parties win with equal probabilities. It follows that, if $x \in X^*$, then there is a complete history after which some party wins with probability at least one half by choosing x. Let

$$\underline{u} \quad = \quad \inf\{u_k(x) \mid x \in X^*\},$$

which is finite, since utilities are bounded. Letting $\hat{x}_k$ denote the core point (and the ideal point of the core voter), suppose that $\underline{u} < u_k(\hat{x}_k)$. By construction, there exist a sequence $\{h_{t_m}\}$ of histories and a sequence $\{x_m\}$ of policies such that (i) for all m, some party, say P_m, wins with probability at least one half by adopting x_m after h_{t_m}, and (ii) $u_k(x_m) \to \underline{u}$. Assume without loss of generality that $P_m = A$ along a subsequence. Indexing that subsequence again by m, we have $P_m = A$ for all m, i.e., B wins with probability less than or equal to one half in the period following each complete history h_{t_m}. We claim that, for high enough m, B can win with probability one after h_{t_m} by deviating to $\hat{x}_k$. If not, then, because the core voter is decisive, we must have

$$(1-\delta)u_k(\hat{x}_k) + \delta v_k(h_{t_m}, x_m, \hat{x}_k, 0) \quad \leq \quad (1-\delta)u_k(x_m) + \delta v_k(h_{t_m}, x_m, \hat{x}_k, 1)$$

for some subsequence (also indexed by m). Since

$$\underline{u} \leq v_k(h_{t_m}, x_m, \hat{x}_k, 0) \quad \text{and} \quad v_k(h_{t_m}, x_m, \hat{x}_k, 1) \leq u_k(\hat{x}_k),$$

this implies

$$(1-\delta)u_k(\hat{x}_k) + \delta\underline{u} \quad \leq \quad (1-\delta)u_k(x_m) + \delta u_k(\hat{x}_k),$$

or equivalently,

$$\frac{u_k(\hat{x}_k) - u_k(x_m)}{u_k(\hat{x}_k) - \underline{u}} \leq \frac{\delta}{1-\delta}.$$

Taking limits, we have

$$\lim_{m\to\infty} \frac{u_k(\hat{x}_k) - u_k(x_m)}{u_k(\hat{x}_k) - \underline{u}} = 1,$$

but $\delta < 1/2$ implies $\delta/(1-\delta) < 1$. This contradiction establishes the claim. Let h_{t_m} be any complete history such that B can win with probability one in the period following. The party's expected discounted payoff is less than or equal to $(1-\delta_P)(1/2)+\delta_P(1)$ if it does not deviate, while it's expected discounted payoff from deviating is at least $(1-\delta_P)(1)+\delta_P(0)$. Since $\delta_P < 1/3$, the deviation is profitable, a contradiction. Therefore, $\underline{u} = u_k(\hat{x}_k)$, and we conclude that $X^* = \{\hat{x}_k\}$. ∎

References

[1] A. Alesina (1988) "Credibility and Policy Convergence in a Two-Party System with Rational Voters," *American Economic Review,* 78: 796-805.

[2] E. Aragones and A. Postlewaite (2000) "Campaign Rhetoric: A Model of Reputation," mimeo.

[3] J. Banks (1995) "Singularity Theory and Core Existence in the Spatial Model," *Journal of Mathematical Economics,* 24: 523-536.

[4] J. Banks and J. Duggan (2006a) "A Dynamic Model of Democratic Elections in Multidimensional Policy Spaces," mimeo.

[5] J. Banks and J. Duggan (2006b) "A Social Choice Lemma on Voting Over Lotteries with Applications to a Class of Dynamic Games," *Social Choice and Welfare,* forthcoming.

[6] D. Baron and E. Kalai (1993) "The Simplest Equilibrium of a Majority Rule Division Game," *Journal of Economic Theory,* 61: 290-301.

[7] D. Bernhardt, E. Hughson, and S. Dubey (2004) "Term Limits and Pork Barrel Politics," *Journal of Public Economics* 88: 2383-2422

[8] D. Black (1958) *The Theory of Committees and Elections,* Cambridge: Cambridge University Press.

[9] G. Cox (1984) "Non-collegial Simple Games and the Nowhere Denseness of the Set of Preference Profiles having a Core," *Social Choice and Welfare,* 1: 159-164.

[10] A. Downs (1957) *An Economic Theory of Democracy,* New York: Harper and Row.

[11] J. Duggan (2000) "Repeated Elections with Asymmetric Information," *Economics and Politics,* 12: 109-136.

[12] J. Fearon (1999) "Electoral Accountability and the Control of Politicians: Selecting Good Types Versus Sanctioning Poor Performance," in Przeworski et al., eds., *Democracy, Accountability, and Representation,* Cambridge: Cambridge University Press.

[13] D. Fudenberg and E. Maskin (1986) "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica,* 54: 533-556.

[14] D. Fudenberg and J. Tirole (1991) *Game Theory,* Cambridge: MIT Press.

[15] G. Kramer (1977) "A Dynamical Model of Political Equilibrium," *Journal of Economic Theory,* 16: 310-334.

[16] M. Le Breton (1987) "On the Core of Voting Games," *Social Choice and Welfare,* 4: 295-305.

[17] R. McKelvey and P. Ordeshook (1985) "Sequential Elections with Limited Information," *American Journal of Political Science,* 29: 480–512.

[18] H. Moulin (1986) "Choosing from a Tournament," *Social Choice and Welfare,* 3: 271-291.

[19] C. Plott (1967) "A Notion of Equilibrium and its Possibility under Majority Rule," *American Economic Review,* 57: 787-806.

[20] A. Rubinstein (1979) "A Note on the Nowhere Denseness of Societies having an Equilibrium under Majority Rule," *Econometrica,* 47: 511-514.

[21] N. Schofield (1983) "Generic Instability of Majority Rule," *Review of Economic Studies,* 50: 695-705.

[22] K. Shotts (2006) "A Signaling Model of Repeated Elections," *Social Choice and Welfare,* forthcoming.