Collective Choice of Fixed-Size Subsets: Plurality Rule, Block Voting, and Arrow's Theorem*

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Abstract

Unlike most of our theoretical analyses, in the real world voters usually do not reveal their entire ranking over the entire set of candidates. Often in fact, the only information revealed by voters is the single candidate for whom the vote is cast. In such a restricted information environment, Goodin and List (2006) offer a "conditional defense" of plurality rule and question whether standard impossibility results can apply. In this paper, we show that a version of Arrow's Theorem can indeed be established in this environment. Our result deals with rules in which each voter cast votes for k candidates and these votes are aggregated into a selection of k candidates. Specifically, our main result is that the only such rules that satisfy both an independence and unanimity condition are dictatorships.

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1 Introduction

A classic topic in the rich and detailed literature on social choice theory is how a group ought to choose a single option out of several possibilities, such as electing a representative out of a group of candidates. In the classic approach to this topic, dating back to Arrow's foundational work (Arrow, 1951), the preference orders of all the individuals in society are aggregated into a single social preference. But it may be that a complete social ranking of the options is not needed; it may suffice to identify the best option for the group. Thus an complementary approach is to focus on social choice functions that take individual preference orders and assign a single social choice.

While these two approaches to understanding voting have been exhaustively studied in the literature, voting systems in the real world rarely require voters to submit complete preference orderings over the entire field of candidates. For example, take plurality rule, which is the most commonly used majoritarian voting system. No matter how many candidates are on the ballot, each voter votes for a single candidate and the candidate with the most votes is chosen. The voter does not submit a full ranking of the candidates, rather the voter simply identifies a single candidates as their choice. In this way, plurality rule and many other commonly used voting systems depart from one important aspect of our theoretical models, namely that the system operates on complete preference rankings of voters. Rather, most voting systems ask that voters mark their ballot with a single choice or an unranked set of choices, and the system counts these votes and generates an election result.

In the literature, the article by Goodin and List (2006) is the only one we are aware of that addresses this point. The authors break a given voting system into two parts: the information about preferences that voters reveal on the ballot and the aggregation of ballots into a social decision. For example, for plurality rule, each voter indicates her top choice on the ballot (but does not give any additional information about her ranking of the other choices) and then these ballots are aggregated by simply choosing the alternative with the most votes. Goodin and List (2006) provide a "conditional defense" of plurality rule: "*If* a society's ballot procedure collects only a single vote from each voter, *then* plurality rule . . . is the uniquely compelling aggregation procedure . . . in the sense that it uniquely satisfies May's well-known minimal conditions on democratic procedures . . . " (Goodin and List, 2006, emphasis in original)

This claim of superiority for plurality rule by Goodin and List is surprising because of the well-known deficiencies of plurality in the standard framework of aggregating full preference orders.[1] These deficiencies arise as a consequence of the fact that plurality rule fails to satisfy Independence of Irrelevant Alternatives (IIA) in the standard framework. Goodin and List argue that this issue is not a concern when voters only select a single alternative on the ballot because "Arrow's theorem . . . cannot be formulated in the present restricted informational environment, as conditions such as

[1]For example, in the standard framework plurality rule may fail to select a Condorcet winner and the selected alternative can depend on the addition or deletion of other alternatives.

. . . IIA . . . are not expressible here.” (Goodin and List, 2006, p. 945)

We show that this claim is incorrect by giving a formulation of IIA for the single-vote ballot environment and proving a version of Arrow’s Theorem in this environment. We establish that in the environment considered by Goodin and List, the only rule that satisfies our version of IIA and a weak unanimity requirement is a dictatorship. This result casts serious doubt on the “conditional defense” of plurality rule given in Goodin and List (2006). While it is true that plurality rule satisfies the version of May’s conditions appropriate to a restricted informational environment, it is seriously deficient as it fails to satisfy the appropriate version of IIA. This is analogous to the place of majority rule in the standard framework: it is the unique rule to satisfy the appropriate version of May’s conditions, but it is seriously deficient as it fails to be transitive in the standard framework.

In fact, we broaden the scope of our analysis by considering not just single-vote ballots, but rather all systems in which each voter chooses a set of alternatives of size k and the voting system selects k winning alternatives. For example, in the block vote electoral system, voters cast as many votes as there are available seats and the candidates with the most votes win, even if they have not managed to secure a majority of the votes. Thus this system is an extension of plurality rule to multimember districts.[2] In our general theorem, we permit k to take a value of one, which shows how we yield the environment of Goodin and List (2006) as a special case. To be precise, we refer to any rule in which voters vote for k (unranked) candidates and the system aggregates these votes into a set of k chosen candidates as a k-rule. Our main result is a version of Arrow’s Theorem that states that a k-rule satisfies appropriate versions of IIA and Pareto if and only if it is a dictatorship.

Of course, the crucial ingredient to our argument is the form of IIA that we specify for the restricted information environment. In the standard environment, IIA states that the social preference between two alternatives should depend only on how individuals rank those two alternatives and should thus be independent of the ranking of other alternatives. Likewise, in the restricted information environment, our version of IIA states that whether or not an alternative is part of the collective choice set should depend only on how individuals judge that specific alternative and should thus be independent of the evaluations of other alternatives.

This way of specifying IIA has precedent in the social choice literature. Specifically, beginning with Kasher and Rubinstein (1997), a number of papers have used this version of IIA in an axiomatic approach to studying group identity (Samet and Schmeidler, 2003; Nicolas, 2007; Çengelci and Sanver, 2010; Ju, 2010; Saporiti, 2012). Our version of Arrow’s Theorem also relates to the classic work of Wilson (1975), who showed how the impossibility result of Arrow can be extended to the aggregation of attributes other than preference orderings. It also relates to the more recent literature on judgement aggregation and the impossibility results presented by List and Pettit (2002), Dietrich

[2]This electoral system is currently is use in some local elections in the United Kingdom; at the national level in places such as Kuwait, Laos, Lebanon, Syria, Tonga and Tuvalu; and was previously used in Jordan, Mongolia, the Philippines and Thailand.

K_1	K_2	K_3	K_4
$\{a, b\}$	$\{b, c\}$	$\{a, b\}$	$\{b, d\}$

Figure 1: Example with $k = 2$

(2007), and Dokow and Holzman (2010). Indeed, it may be possible to deduce a version of our result from the deep theorem in Dokow and Holzman (2010) but at the cost of considerable complexity. In contrast, the self-contained version given here clearly links our arguments to more traditional proofs of Arrow's Theorem.

2 Notation and Axioms

Consider a society N composed of $n \geq 3$ *individuals*, so that $N = \{1, \ldots, n\}$, and a set A composed of m *alternatives*, with $m \geq 3$. For $k \in \{1, \ldots, m-1\}$, we let $\mathcal{A}_k$ be the collection of subsets of A of size k.

Fix a value of $k \in \{1, \ldots, m-1\}$. We think of k as the number of (distinct) votes cast by each individual in society. More broadly, we can think of each individual as having a set of k alternatives that is viewed as the best choice by that individual. Formally, for each individual i, we assign $K_i \in \mathcal{A}_k$ to individual i, and we define a profile as $K = (K_1, \ldots, K_n)$. We refer to K_i as individual i's *chosen set*. Note that the k alternatives in K_i are not ranked relative to each other—the only information available for each individual is which k alternatives are in K_i (and which $m - k$ alternatives are not).

We are interested in aggregating profiles into a single set of size k that is viewed as the collective choice of society. Again, we can view this as the procedure to fill k seats in a multimember district, for example. Formally, an k-set aggregation rule is a mapping $F : \mathcal{A}_k \times \cdots \times \mathcal{A}_k \to \mathcal{A}_k$, that assigns to each profile K a set $F(K) \in \mathcal{A}_k$. We refer to $F(K)$ as the *collective choice set*.

As an example, suppose the set of alternatives is $A = \{a, b, c, d, e\}$, society is composed of four individuals, and $k = 2$ so that each individual votes for two alternatives and the aggregation rule selects two of the alternatives as the collective choice. In particular, suppose the profile $K = (K_1, \ldots, K_n)$ is given by Figure 1. As illustrated in the figure, voter 1 chooses the two alternatives a and b, voter 2 chooses the two alternatives b and c and so on. As an example of a 2-set aggregation rule, consider the block voting rule in which the two alternatives chosen by the most alternatives are selected. In the example depicted in Figure 1, this rule selects the two alternatives a and b.

We next define the axiomatic properties that will be used in our analysis.[3] First, an aggregation rule satisfies IIA if each alternative is included as a collective choice purely as a function of how each

[3]For more discussion of these axioms, see Kasher and Rubinstein (1997) and Saporiti (2012).

K_1	K_2	K_3	K_4
$\{a, c\}$	$\{d, e\}$	$\{a, b\}$	$\{b, c\}$

Figure 2: Example for IIA

individual in society views that particular alternative. In other words, IIA requires that whether or not the alternative is included in the collective choice is independent of how other alternatives are viewed. Formally we state the following axiom:

IIA A k-set aggregation rule F satisfies Independence of Irrelevant Alternatives if, for all $x \in A$ and for all profiles $(K_1, \ldots, K_n)$ and $(K'_1, \ldots, K'_n)$, such that $x \in K_i$ if and only if $x \in K'_i$ for all $i \in N$, we have $x \in F(K_1, \ldots, K_n)$ if and only if $x \in F(K'_1, \ldots, K'_n)$.

Note that with only two alternatives, the IIA axiom is always trivially satisfied. Thus, we consider situations in which there are three or more alternatives.

As an example of the IIA axiom, suppose that for the profile given in Figure 1 the collective choice is the set $\{a, b\}$ and now consider the profile given in Figure 2. As alternative a is included in the chosen set in this profile by exactly the same individuals as in Figure 1, IIA requires that a be included in the collective choice set of the profile in Figure 2. Note however, that for an alternative such as b which is included in the chosen set by different individuals in the two profiles, IIA says nothing about whether b ought to be in the collective choice set of the second profile.

Next, an aggregation rule satisfies Pareto if unanimous agreement about the status of an alternative generates the same status in the collective choice. Specifically, if every individual either includes or excludes some alternative in their chosen set, then so must the collective choice set. Formally, we have:

Pareto A k-set aggregation rule F satisfies Pareto if, for all $x \in A$, and for all profiles $(K_1, \ldots, K_n)$ such that $x \in K_i$ for all $i \in N$, we have $x \in F(K_1, \ldots, K_n)$, and for all profiles $(K_1, \ldots, K_n)$ such that $x \notin K_i$ for all $i \in N$, we have $x \notin F(K_1, \ldots, K_n)$.

Saporiti (2012) defines a weaker version of this axiom as follows.

Weak Pareto A k-set aggregation rule F satisfies Pareto if, for all $x \in A$, and all profiles $(K_1, \ldots, K_n)$, if $x \in K_i$ for all $i \in N$, we have $x \in F(K_1, \ldots, K_n)$.

Thus, Weak Pareto only requires that an alternative be included in the collective choice if there is unanimous agreement about this alternative. It places no requirement on how unanimous agreement on exclusion is handled by the aggregation rule. As we prove in the next section, Weak Pareto and IIA together implies Pareto.

As an example of the two versions of Pareto, again consider the profile given in Figure 1. As alternative b is present in every individual's chosen set and alternative e is present in nobody's chosen set, Pareto requires that the collective choice set for this profile include b and exclude e. Weak Pareto, on the other hand, only requires that b be included in the collective choice set, but places no restriction on whether e is included or excluded.

Finally, we define a particular aggregation rule, known as *dictatorship*. Formally, an aggregation rule F is a dictatorship if there exists an individual $i \in N$ such that $F(K_1, \ldots, K_n) = K_i$ for all profiles $(K_1, \ldots, K_n)$. That is, an individual is a dictator if the collective choice set is always identical to the individual's chosen set.

3 Results

In this section we prove our main result. Our approach mirrors the standard proof of Arrow's Theorem and thus illustrates the connection between impossibility in the standard framework and our impossibility result in the restricted information environment. We proceed by establishing a sequence of lemmas and conclude by stating our main theorem and show how it follows from the lemmas.

Our first lemma is a perhaps surprising consequence of two of our axioms. It states that, assuming IIA and Weak Pareto, the collective choice set for a profile is always equal to some individual's chosen set.[4]

Lemma 1. *Suppose F is a k-set aggregation rule with $k \in \{1, \ldots, m-1\}$ that satisfies IIA and Weak Pareto. Then for all profiles $(K_1, \ldots, K_n)$, $F(K_1, \ldots, K_n) = K_i$ for some $i \in N$.*

Proof. Suppose F satisfies IIA and Weak Pareto, but there exists some profile $(K_1, \ldots, K_n)$ such that $F(K_1, \ldots, K_n) \neq K_i$ for all $i \in N$. Then for each $i \in N$, there exists an alternative $\hat{x}_i \in K_i$ such that $\hat{x}_i \neq F(K_1, \ldots, K_n)$. Construct a new profile $(K'_1, \ldots, K'_n)$ as follows. For each $i \in N$, if $\hat{x}_1 \in K_i$ then $K'_i = K_i$ and if $\hat{x}_1 \notin K_i$ then we replace $\hat{x}_i$ with $\hat{x}_1$ in K'_i. That is, if $\hat{x}_1 \notin K_i$ then $K'_i = \{\hat{x}_1\} \cup (K_i \setminus \{\hat{x}_i\})$. By IIA, all alternatives in $F(K_1, \ldots, K_n)$ are also in $F(K'_1, \ldots, K'_n)$ and by Weak Pareto, $\hat{x}_1 \in F(K'_1, \ldots, K'_n)$. But then $F(K'_1, \ldots, K'_n)$ contains at least $k+1$ alternatives, which is impossible. This proves the lemma. $\square$

It should be emphasized that this lemma does not imply that the rule is a dictatorship because, while the collective choice set must be some individual's chosen set, two different profiles could have collective choice sets that correspond to two different individuals' chosen sets.

An immediate consequence of this lemma is that if $x \notin K_i$ for all $i \in N$, then $x \notin F(K_1, \ldots, K_n)$. We thus have the following corollary:

[4]This is similar to the axiom of Individual Support discussed by Fey (2004).

Corollary 1. *If a k-set aggregation rule with $k \in \{1, \ldots, m-1\}$ satisfies IIA and Weak Pareto, then it satisfies Pareto.*

We can make our work somewhat easier by noting that for $k \in \{1, \ldots, m-1\}$, the problem of aggregating n individual sets of size k into a collective set of size k consistent with IIA and Pareto has a natural dual problem of aggregating n sets of size $m-k$ into a collective set of size $m-k$ consistent with the two axioms. That is, the problem of choosing a collective set of size k as a function of individual sets of size k is equivalent to specifying a set of $m-k$ alternatives that should not be the collective choice as a function of the $m-k$ alternatives that each individual does not choose.

We state this in the next lemma:

Lemma 2. *Suppose $k \in \{1, \ldots, m-1\}$. A k-set aggregation rule F satisfies IIA and Weak Pareto if and only if the $m-k$-set aggregation rule $\bar{F}$ given by $\bar{F}(\bar{K}_1, \ldots, \bar{K}_n) = A \setminus F(A \setminus \bar{K}_1, \ldots, A \setminus \bar{K}_n)$ satisfies IIA and Weak Pareto.*

Proof. Suppose a k-set aggregation rule F satisfies IIA and Weak Pareto. Then by Corollary 1, it satisfies Pareto. Define a $m-k$-set aggregation rule $\bar{F}$ by $\bar{F}(\bar{K}_1, \ldots, \bar{K}_n) = A \setminus F(A \setminus \bar{K}_1, \ldots, A \setminus \bar{K}_n)$. Suppose there is $x \in A$ such that $x \in \bar{K}_i$ for all $i \in N$. Then $x \notin A \setminus \bar{K}_i$ for all $i \in N$ and so by Pareto, $x \notin F(A \setminus \bar{K}_1, \ldots, A \setminus \bar{K}_n)$. But then $x \in \bar{F}(\bar{K}_1, \ldots, \bar{K}_n)$ and $\bar{F}$ satisfies Weak Pareto. A similar argument show that $\bar{F}$ satisfies IIA. □

A consequence of this lemma is that, without loss of generality, we can take $k \leq m/2$ because for all $k > m/2$, we can instead work with the dual problem with $m-k < m/2$. Thus, from this point forward we suppose that $k \leq m/2$.

We say a group $G \subseteq N$ is semi-decisive for some $K \in \mathcal{A}_k$ if there is a profile $(K_1, \ldots, K_n)$ satisfying $K_i = K$ for all $i \in G$ and $K_i \cap K = \emptyset$ for all $i \notin G$ such that $F(K_1, \ldots, K_n) = K$. That is, when a group is semi-decisive, if all members of the group have identical chosen sets and the chosen sets of all individuals outside the group have no alternatives in common with the chosen set of the group members, then the group as acts as a group dictator—the collective choice set is equal to the common chosen set of the group members. Note that if a a group $G \subseteq N$ is semi-decisive for some $K \in \mathcal{A}_k$, then by IIA, $F(K_1, \ldots, K_n) = K$ for every profile $(K_1, \ldots, K_n)$ satisfying $K_i = K$ for all $i \in G$ and $K_i \cap K = \emptyset$ for all $i \notin G$.

In a similar fashion, we say a group $G \subseteq N$ is decisive if, for every $K \in \mathcal{A}_k$ and for every profile $(K_1, \ldots, K_n)$ satisfying $K_i = K$ for all $i \in G$, $F(K_1, \ldots, K_n) = K$. In other words, if a group is decisive, then whenever all members of the group have identical chosen sets, the group as acts as a group dictator—the collective choice set is equal to the common chosen set of the group members. Clearly if F satisfies Weak Pareto, then the group $G = N$ consisting of the entire society is decisive.

These definitions mirror those made in the classic proof of Arrow's Theorem. Likewise, the next two lemmas mirror Sen's "field expansion lemma." The first of these two lemmas states that if a group is semi-decisive for some chosen set, it is semi-decisive for all chosen sets.

Lemma 3. *Suppose F is a k-set aggregation rule with $k \in \{1, \ldots, m-1\}$ that satisfies IIA and Weak Pareto. Then if there exists some $\hat{K} \in \mathcal{A}_k$ such that a group $G \subseteq N$ is semi-decisive for $\hat{K}$, then G is semi-decisive for every $K \in \mathcal{A}_k$.*

Proof. Suppose F satisfies IIA and Weak Pareto and suppose there exists some $\hat{K} \in \mathcal{A}_k$ such that a group G is semi-decisive for $\hat{K}$. Recall that by IIA, $F(K_1, \ldots, K_n) = \hat{K}$ for every profile $(K_1, \ldots, K_n)$ satisfying $K_i = \hat{K}$ for all $i \in G$ and $K_i \cap \hat{K} = \emptyset$ for all $i \notin G$.

We consider two cases. The first case is $k = 1$. In this case, $\hat{K} = \{\hat{x}\}$ for some $\hat{x} \in A$. Now pick $y, z \in A$ such that $\hat{x}$, y, and z are distinct. This is possible because $m \geq 3$. Define the profile $(K_1, \ldots, K_n)$ by $K_i = \{y\}$ for all $i \in G$ and $K_i \neq \{y\}$ for all $i \notin G$. In order to show that $F(K_1, \ldots, K_n) = \{y\}$, suppose not. Then by IIA, the profile $(K'_1, \ldots, K'_n)$ defined by $K'_i = \{y\}$ for all $i \in G$ and $K'_i = \{z\}$ for all $i \notin G$ must also have $F(K'_1, \ldots, K'_n) \neq \{y\}$. But then Lemma 1 requires that $F(K'_1, \ldots, K'_n) = \{z\}$. Finally, consider the profile $(K''_1, \ldots, K''_n)$ defined by $K''_i = \{\hat{x}\}$ for all $i \in G$ and $K''_i = \{z\}$ for all $i \notin G$. Comparing this profile to the profile $(K'_1, \ldots, K'_n)$, IIA requires that $F(K''_1, \ldots, K''_n) = \{z\}$. But as G is semi-decisive for $\{\hat{x}\}$, we must have $F(K''_1, \ldots, K''_n) = \{\hat{x}\}$. This contradiction shows that G is semi-decisive for every singleton $\{x\}$ in A.

The second case we consider is $k \geq 2$. We first show that G is semi-decisive for every $K \in \mathcal{A}_k$ such that $K \cap \hat{K} \neq \emptyset$. Let $\hat{x} \in K \cap \hat{K}$ and let the profile $(K'_1, \ldots, K'_n)$ satisfy $K'_i = K$ for all $i \in G$ and $K'_i \cap K = \emptyset$ for all $i \notin G$. Therefore $\hat{x} \in K'_i$ if and only if $i \in G$ and so by IIA, $\hat{x} \in F(K'_1, \ldots, K'_n)$. Thus it follows from Lemma 1 that $F(K'_1, \ldots, K'_n) = K$, and so G is semi-decisive for K.

Next we show that G is semi-decisive for every $K \in \mathcal{A}_k$. Let the profile $(K'_1, \ldots, K'_n)$ satisfy $K'_i = K$ for all $i \in G$ and $K'_i \cap K = \emptyset$ for all $i \notin G$. Pick arbitrary alternatives $x \in K$ and $\hat{x} \in \hat{K}$ and pick K'' such that $\{x, \hat{x}\} \subseteq K''$. This is possible because $k \geq 2$. As $K'' \cap \hat{K} \neq \emptyset$, G is semi-decisive for K''. It follows that $F(K''_1, \ldots, K''_n) = K''$ for a profile $(K''_1, \ldots, K''_n)$ that satisfies $K''_i = K''$ for all $i \in G$ and $K''_i \cap K'' = \emptyset$ for all $i \notin G$. This means that $x \in F(K''_1, \ldots, K''_n)$, so it follows by IIA, we must have $x \in (K'_1, \ldots, K'_n)$. Finally, it follows from Lemma 1 that $F(K'_1, \ldots, K'_n) = K$. Therefore G is semi-decisive for every K. □

The second of these two lemmas states that if a group is semi-decisive for all chosen sets, then it is decisive.

Lemma 4. *Suppose F is a k-set aggregation rule with $k \in \{1, \ldots, m-1\}$ that satisfies IIA and Weak Pareto. Then if a group $G \subseteq N$ is semi-decisive for every $K \in \mathcal{A}_k$, then G is decisive.*

Proof. Suppose F satisfies IIA and Weak Pareto and suppose G is semi-decisive for every $K \in \mathcal{A}_k$. By Lemma 2, we can take $k \leq m/2$. To show that G is decisive, suppose not. That is, suppose that for some profile $(K_1, \ldots, K_n)$ with $K_i = K$ for all $i \in G$ we have $F(K_1, \ldots, K_n) \neq K$. By Lemma 1, there exists $j \in N$ such that $F(K_1, \ldots, K_n) = K_j$. Therefore, $K_j \neq K$ and so $j \notin G$. This implies there exists $\bar{x} \in K \setminus K_j$ and $x_j \in K_j \setminus K$. Obviously, $\bar{x} \neq x_j$.

Construct a new profile $(K'_1, \ldots, K'_n)$ as follows. For individual j, $K'_j = K_j$, and for all $i \in G$, $K'_i = K_i$. For all $i \in N \setminus (G \cup \{j\})$, we require that $x_j \in K'_i$ if and only $x_j \in K_i$ be satisfied and otherwise all remaining alternatives in K'_i can be chosen arbitrarily from $A \setminus (\{x_j, \bar{x}\})$. As $k \leq m/2$, this is always possible. As $x_j \in K'_i$ if and only $x_j \in K_i$ for all $i \in N$ and $x_j \in F(K_1, \ldots, K_n)$, then by IIA, $x_j \in F(K'_1, \ldots, K'_n)$. But as $G \subseteq N$ is semi-decisive for K, there is a profile $(K''_1, \ldots, K''_n)$ satisfying $K''_i = K$ for all $i \in G$ and $K''_i \cap K = \emptyset$ for all $i \notin G$ such that $F(K''_1, \ldots, K''_n) = K$. As $\bar{x} \in K'_i$ if and only if $i \in G$, then by IIA applied to profiles $(K'_1, \ldots, K'_n)$ and $(K''_1, \ldots, K''_n)$, $\bar{x} \in F(K'_1, \ldots, K'_n)$. Thus it follows from Lemma 1 that $F(K'_1, \ldots, K'_n) = K$, but this contradicts $x_j \in F(K'_1, \ldots, K'_n)$. We conclude that for every profile $(K_1, \ldots, K_n)$ satisfying $K_i = K$ for all $i \in G$ we must have $F(K_1, \ldots, K_n) = K$. □

Combining this lemma with Lemma 3 allows us to conclude that if a group is semi-decisive for some chosen set, then it is decisive. Our next lemma mirrors the "group contraction lemma" of Sen. It states that a decisive group must contain a strictly smaller decisive group.

Lemma 5. *Suppose F is a k-set aggregation rule with $k \in \{1, \ldots, m-1\}$ that satisfies IIA and Weak Pareto. If a group G with $\#G \geq 2$ is decisive, then a non-empty proper subset of G is decisive.*

Proof. Suppose F satisfies IIA and Weak Pareto and suppose G with $\#G \geq 2$ is decisive. By Lemma 2, we can take $k \leq m/2$. Partition G into two non-empty groups G_1 and G_2. Pick $K^1 \in \mathcal{A}_k$ and $K^2 \in \mathcal{A}_k$ that differ by one element. That is, $K^1 \setminus K^2 = \{x_1\}$ and $K^2 \setminus K^1 = \{x_2\}$. Let the profile $(K_1, \ldots, K_n)$ satisfy $K_i = K^1$ for all $i \in G_1$, $K_i = K^2$ for all $i \in G_2$, and $K_i \cap \{x_1, x_2\} = \emptyset$ for all $i \notin G$. This is possible because $k \leq m/2$. By Lemma 1, $F(K_1, \ldots, K_n) = K_i$ for some $i \in N$. We first establish that $F(K_1, \ldots, K_n)$ equals K^1 or K^2. To see this, suppose not. That is, suppose $F(K_1, \ldots, K_n) = K_j$ for some $j \notin G$. As $K_j \neq K^1$ by construction, there exists $x_j \in K_j \setminus K^1$. But consider a new profile $(K'_1, \ldots, K'_n)$ that satisfies $K'_i = K^1$ for all $i \in G$, and $K'_i = K_i$ for all $i \notin G$. As $x_j \in K_i$ if and only if $x_j \in K'_i$ and $x_j \in F(K_1, \ldots, K_n)$, by IIA we have $x_j \in F(K'_1, \ldots, K'_n)$. However, as G is decisive, $F(K_1, \ldots, K_n) = K^1$, so $x_j \notin F(K'_1, \ldots, K'_n)$. This contradiction establishes that $F(K_1, \ldots, K_n) \in \{K^1, K^2\}$.

To continue, assume without loss of generality that $F(K_1, \ldots, K_n) = K^1$. We will show that this implies that G_1 is decisive. Consider a profile $(K''_1, \ldots, K''_n)$ satisfying $K''_i = K^1$ for all $i \in G_1$ and $K''_i \cap K^1 = \emptyset$ for all $i \notin G_1$. As $x_1 \in K''_1$ if and only if $x_1 \in K_1$, it follows from IIA that

$x_1 \in F(K_1'', \ldots, K_n'')$. Then it follows from Lemma 1 that $F(K_1'', \ldots, K_n'') = K^1$ and therefore G_1 is semi-decisive for K^1. Applying Lemmas 3 and 4, G_1 is decisive. This proves the lemma. $\square$

We are now ready to state and prove our version of Arrow's Theorem for k-set aggregation rules. This is our main result.

Theorem 1. *A k-set aggregation rule with $k \in \{1, \ldots, m-1\}$ satisfies IIA and Weak Pareto if and only if it is a dictatorship.*

Proof. Clearly if F is a dictatorship, then F satisfies IIA and Weak Pareto. For the converse, suppose F satisfies IIA and Weak Pareto. By Weak Pareto, N is decisive. Therefore there exists a decisive group of minimal size, call it G^*. If $\#G^* \geq 2$, then by Lemma 5 there is a non-empty proper subset of G^* which is decisive. But this contradicts the fact that G^* is a decisive set of minimal size, so $\#G^* = 1$. We conclude that F is a dictatorship. $\square$

4 Conclusion

In this paper we have considered voting systems in which each voters casts k votes in order to fill k seats. Examples of such systems in the real world include plurality, in which case $k = 1$, and block voting, in which case $k > 1$. We have shown that there is no such system that satisfies IIA and Pareto, other than dictatorship. In particular, plurality rule must violate our version of IIA, just as it does in the standard environment.

There are several other electoral systems that our framework does not cover. As described by Cox (1990), limited vote systems are similar to block voting systems except voters have fewer votes than there are seats to be filled. Cumulative voting describes systems in which may cast more than one vote for a single candidate. Finally, some block voting systems allow voters to partially abstain and not use all of their available votes. We suspect that impossibility results similar to Theorem 1 will hold in these systems as well, but we leave such questions to future work.

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