Lectures in Quantitative International Relations
Introduction to Maximum Likelihood

Kevin A. Clarke
University of Rochester

Dublin City University, May 2007
Lectures in this Series
Introduction to Maximum Likelihood Estimation
Lectures in this Series

1. Introduction to Maximum Likelihood Estimation
2. Common ML Models Used in International Relations
Lectures in this Series

1. Introduction to Maximum Likelihood Estimation
2. Common ML Models Used in International Relations
3. Comparative Theory Testing
Lectures in this Series

1. Introduction to Maximum Likelihood Estimation
2. Common ML Models Used in International Relations
3. Comparative Theory Testing
4. Choosing a Specification
Course Web Site

http://www.rochester.edu/college/psc/clarke/Dublin/Dublin.html
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Includes the following:

- syllabus
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Includes the following:

- syllabus
- exercises
- data sets
- codebooks
- relevant articles
- software resources
Why the Focus on Maximum Likelihood?

- Crescenzi (2007). “Reputation and Interstate Conflict.” *AJPS.*
- Danilovic and Clare (2007). “The Kantian Liberal Peace (Revisited)” *AJPS.*
What are our Goals?

What should you get out of these lectures?
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- intuition
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- intuition
- an idea of what’s out there
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- intuition
- an idea of what’s out there
- a basis for learning new techniques
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What should you get out of these lectures?

- intuition
- an idea of what’s out there
- a basis for learning new techniques
- help in reading the literature
Overview of Lecture 1

The Logic of Political Survival
- The Model
- Explaining the Bias
Overview of Lecture 1

1. The Logic of Political Survival
   - The Model
   - Explaining the Bias

2. Maximum Likelihood Theory
   - Intuition
   - Fundamentals
   - MLE and Regression
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3. R
The Logic of Political Survival

R

The Model

Explaining the Bias

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Lectures in Quantitative International Relations
The Selectorate Model

- Leaders want to survive.
The Selectorate Model

- Leaders want to survive.
- Selectorate \((S)\) and minimum winning coalition \((W)\).
The Selectorate Model

- Leaders want to survive.
- Selectorate ($S$) and minimum winning coalition ($W$).
- Public vs. private goods.
The Selectorate Model

- Leaders want to survive.
- Selectorate \((S)\) and minimum winning coalition \((W)\).
- Public vs. private goods.
- Loyalty norm \((W/S)\).
An Implication of the Model

Direct implication of the model:

- Institutions that call for larger winning coalitions should be associated with more effort to produce public goods.

Thus, $W$ should be positively associated with public expenditures (measured as a proportion of GDP).
# Expenditures

<table>
<thead>
<tr>
<th>Variable</th>
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<tbody>
<tr>
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What are democracy residuals and ln(GDP) residuals?
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What happens when we substitute the original variables back into the regression?
### Expenditures

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They actually run this instead:

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where \( \hat{\nu} \) are the residuals from the auxiliary regression,

\[ d = \gamma_0 + w\gamma_1 + \nu. \]
Kennedy’s Ballentine
Kennedy’s Ballentine

\[ E \]

\[ W \]

\[ D \]
Kennedy’s Ballentine
Intuition

Kennedy’s Ballentine
Intuition

Kennedy’s Ballentine
Kennedy’s Ballentine
Kennedy’s Ballentine
The Estimated Coefficient on $W$

Given the regression

$$y = \beta_0 + w \beta_1 + \nu \beta_2 + \epsilon,$$

the expected value of this estimator, assuming $E[\epsilon] = 0$, is

$$E[\hat{\beta}_1] = \beta_1 + (w'w)^{-1}w'd\beta_2,$$
This Is Omitted Variable Bias

Compare the expectation from the last frame

\[ E[\hat{\beta}_1] = \beta_1 + (w'w)^{-1}w'd\beta_2 \]

to the standard omitted variable bias result

\[ E[\hat{\beta}_1] = \beta_1 + (x'_1x_1)^{-1}x'_1x_2\beta_2. \]
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In most of their regressions, BdM and coauthors estimated the effect of \( w \) as if \( d \) were not in the equations.
Implications

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- Last, but not least...
- Know what you are doing.
With that in mind, let’s turn to learning maximum likelihood theory. Remember, if some of the math is unfamiliar to you, try and get the larger picture.
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Let’s start with some intuition.
Consider a sample of great power militarized disputes. If we know the probability of escalation, the binomial formula tells us the probability of, say, 3 disputes out of 10 escalating.
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If the probability of escalation is $\theta = 0.3$, 

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\Pr(x|\theta, N) = \binom{N}{x} \theta^x (1-\theta)^{N-x}
\]

\[
\frac{10}{3} \times 0.3^3 \times (1-0.3)^{10-3} = 0.27
\]
The Binomial Probability

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- If $\theta = 0.7$, $Pr(3|0.7, 10) = 0.009$
- If $\theta = 0.9$, $Pr(3|0.9, 10) \approx 0$
The Maximum Likelihood Estimator

\( \theta = 0.3 \) is the value that makes observing 3 escalations out of 10 militarized disputes most likely.
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The ML estimate is the value of the parameter that makes the observed data most likely.
The Logic of Political Survival

Maximum Likelihood Theory

R

Intuition

Fundamentals

MLE and Regression

The Likelihood Function

\[ \Pr(3|\theta,10) \]

Clarke

Lectures in Quantitative International Relations
Definitions and Notation

Random variable $X$

Realization $X = x$

Sample $X = (X_1, X_2, \ldots, X_n)$

Realization $X = x$

Random sample: a sample where the random variables are independent and identically distributed.

Joint density of the sample: given a random sample, it is the product of the individual densities,

$$f(x; \theta) = f(x_1, x_2, \ldots, x_n; \theta) = \prod_{k=1}^{n} f_k(x_k; \theta).$$
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$$f(x; \theta) = f(x_1, x_2, \ldots, x_n; \theta) = \prod_{k=1}^{n} f_k(x_k; \theta).$$
Let \( f(x; \theta) \) be the joint density of the sample.

The *likelihood function* is

\[
L(\theta; x) = f(x; \theta).
\]

Isn’t that the joint density of the sample? What’s the difference?
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$f(x; \theta) \rightarrow \theta$ is fixed and $x$ is variable.
$L(\theta; x) \rightarrow x$ is the observed sample point and $\theta$ varies over all possible parameter values.
If $X$ is discrete, $L(\theta; x) = \Pr(X = x)$. 

This tells us that the sample actually observed is more likely to have occurred if $\theta = \theta_1$ than if $\theta = \theta_2$. So $\theta_1$ is a more plausible value for the true value of $\theta$ than is $\theta_2$. 
Comparing Likelihoods

If $X$ is discrete, $L(\theta; x) = \Pr(X = x)$.

If we compare the likelihood function at 2 different values of $\theta$, $\theta_1$ and $\theta_2$, we may find that

$$\Pr_{\theta_1}(X = x) = L(\theta_1; x) > L(\theta_2; x) = \Pr_{\theta_2}(X = x).$$
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So \( \theta_1 \) is a more plausible value for the true value of \( \theta \) than is \( \theta_2 \).
Maximum Likelihood Estimators

For each sample point \( x \), let \( \hat{\theta}(x) \) be a parameter value at which \( L(\theta; x) \) attains its maximum as a function of \( \theta \), with \( x \) held fixed. The maximum likelihood estimator is

\[
\hat{\theta}\left(\hat{\theta}(x); x\right) = \max_{\hat{\theta}} L(\theta; x)
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Remember the intuition... the MLE is the parameter point for which the observed sample is most likely.
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Finding the MLE

We find the MLE by finding the *maximum* of the likelihood function.
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Why do we care about the derivative of the function?

Because, loosely speaking, the derivative at a point on the curve is the slope of the line that is tangent to the curve at the point. And the line that is tangent to the maximum must have a particular slope....
The Tangent Line

![Graph showing the tangent line at a point on a curve]

- The graph represents the probability function $Pr(3|\theta,10)$.
- The x-axis is labeled Theta, ranging from 0.0 to 1.0.
- The y-axis shows the probability values ranging from 0.00 to 0.25.

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Lectures in Quantitative International Relations
So, the maximization problem to solve is

$$\max_{\theta} L(\theta; x),$$

and we solve it by setting the derivative to zero:

$$\frac{dL(\theta; x)}{d\theta} = 0,$$

and solving for $\theta$. 
Example 1

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Now, it turns out that it is easier to maximize likelihood functions after taking the log of the likelihood function.
We can do this because the log is a monotonic transformation, which has its maximum in the same place.
But why should we take the log?
The Log-likelihood Function

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Remember that the likelihood function is a product.
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1. It is easier to take expectations and variances of sums as opposed to products.
The Log-likelihood Function

But why should we take the log?

Remember that the likelihood function is a product.

1. It is easier to take expectations and variances of sums as opposed to products.
2. It makes it possible for the computer to deal with large numbers of observations.
Let’s Explain Point 2

In a simple case, the likelihood is equal to a probability:

$$L(\theta; x_j) = \Pr(\text{we would observe } x_j).$$
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Let’s say we have 2000 observations, and each probability is around 0.5 The probability of the data set would be

\[ 0.5^{2000} = 2 \times 10^{-603} \]
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\[ 0.5^{2000} = 2 \times 10^{-603} \]

This value is too small to be computed by most computers. But if we take the log....

\[ \ln(0.5^{2000}) = 2000 \times \ln(0.5) \approx 2000 \times -0.6931 = -1386.2. \]
Example 1 continued

The likelihood function is

\[ L(\theta; x) = \prod_{i=1}^{n} \theta^{x_i} (1 - \theta)^{1-x_i}. \]
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\[ L(\theta; \mathbf{x}) = \prod_{i=1}^{n} \theta^{x_i} (1 - \theta)^{1-x_i}. \]

Taking the log,

\[
\ln L(\theta; \mathbf{x}) = \sum_{i=1}^{n} \left[ x_i \ln \theta + (1 - x_i) \ln(1 - \theta) \right]
\]

\[
= \left( \sum_{i=1}^{n} x_i \right) \ln \theta + \left( \sum_{i=1}^{n} [1 - x_i] \right) \ln(1 - \theta)
\]
Example 1 continued

Next we take the log and set it equal to 0,

\[
\frac{d \ln L(\theta; x)}{d \theta} = \left( \sum_{i=1}^{n} x_i \right) \frac{1}{\theta} - \left( \sum_{i=1}^{n} [1 - x_i] \right) \frac{1}{1 - \theta} = 0
\]

\[
(1 - \theta) \sum_{i=1}^{n} x_i - \theta \left( \sum_{i=1}^{n} [1 - x_i] \right) = 0
\]

\[
\sum_{i=1}^{n} x_i - \theta \sum_{i=1}^{n} x_i - n\theta + \theta \sum_{i=1}^{n} x_i = 0
\]

\[
\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n}
\]
Example 2

Suppose that we have a random sample from a Poisson distribution. The likelihood is

\[ L(\lambda; x) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}. \]
Example 2

Suppose that we have a random sample from a Poisson distribution. The likelihood is

$$L(\lambda; \mathbf{x}) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}.$$  

Taking the log,

$$\ln L(\lambda; \mathbf{x}) = \sum_{i=1}^{n} [-\lambda + x_i \ln \lambda - \ln x_i!] .$$
Example 2: the derivative

\[
\frac{d \ln L(\lambda; x)}{d\lambda} = \sum_{i=1}^{n} \left( -1 + \frac{x_i}{\lambda} \right) = 0
\]

\[
\sum_{i=1}^{n} (-1) + \sum_{i=1}^{n} \frac{x_i}{\lambda} = 0
\]

\[
\sum_{i=1}^{n} \frac{x_i}{\lambda} = n
\]

\[
\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i
\]
Example 3

Suppose that we have a random sample from the following density:

\[ f(x; \theta) = \begin{cases} 
\theta x_i^{\theta - 1} & \text{for } 0 < x_i < 1 \\
0 & \text{otherwise}
\end{cases} \]

Find the MLE of \( \theta \) \((\theta > 0)\).
Example 3: solution

\[ \ln L(\theta; x) = \sum_{i=1}^{n} [\ln \theta + (\theta - 1) \ln x_i]. \]
Example 3: solution

\[ \ln L(\theta; x) = \sum_{i=1}^{n} [\ln \theta + (\theta - 1) \ln x_i]. \]

\[ \frac{d \ln L(\theta; x)}{d\theta} = \sum_{i=1}^{n} \left[ \frac{1}{\theta} + \ln x_i \right] = 0 \]

\[ \sum_{i=1}^{n} \frac{1}{\theta} = - \sum_{i=1}^{n} \ln x_i \]

\[ \hat{\theta} = - \frac{n}{\sum_{i=1}^{n} \ln x_i} \]
Technical Point

A first derivative that equals 0 is a *necessary*, but not *sufficient* condition for a maximum.

How can we distinguish between a maximum and a minimum?
A first derivative that equals 0 is a \textit{necessary}, but not \textit{sufficient} condition for a maximum.

How can we distinguish between a maximum and a minimum?

A summit is reached when we ascend and then descend. Thus, the derivative must be declining, going from positive to zero to negative.

That is, the second derivative (the derivative of the derivative) must be negative.
Example 3: A Maximum?

The log-likelihood function:

\[ g(\theta) = \ln L(\theta; x) = \sum_{i=1}^{n} [\ln \theta + (\theta - 1) \ln x_i]. \]
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The log-likelihood function:

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The first derivative:

\[ g'(\theta) = \sum_{i=1}^{n} \left[ \frac{1}{\theta} + \ln x_i \right] \]
Example 3: A Maximum?

The log-likelihood function:

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The first derivative:

\[ g'(\theta) = \sum_{i=1}^{n} \left[ \frac{1}{\theta} + \ln x_i \right] \]

The second derivative:

\[ g''(\theta) = \sum_{i=1}^{n} \left[ -\frac{1}{\theta^2} \right] \]
Simple Regression in MLE

The model is

\[ y_i = \alpha + \beta x_i + \epsilon_i, \quad \epsilon_i \sim \text{IID } N(0, \sigma^2). \]
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\[ y_i = \alpha + \beta x_i + \epsilon_i, \quad \epsilon_i \sim \text{IID } N(0, \sigma^2). \]

Since the \( y_i \) are independently and normally distributed with means \( \alpha + \beta x_i \) and common variance, \( \sigma^2 \), the density of each observation follows a normal distribution:

\[ f(x_i) = \left( \frac{1}{2\pi\sigma^2} \right)^{1/2} \exp \left[ -\frac{1}{2\sigma^2} (x_i - \mu)^2 \right] \]

or

\[ f(y_i) = \left( \frac{1}{2\pi\sigma^2} \right)^{1/2} \exp \left[ -\frac{1}{2\sigma^2} (y_i - \alpha - \beta x_i)^2 \right]. \]
Simple Regression in MLE

The likelihood is therefore the product over the $n$ observations:

$$L(y) = \prod_{i=1}^{n} \left( \frac{1}{2\pi\sigma^2} \right)^{1/2} \exp \left[ -\frac{1}{2\sigma^2} (y_i - \alpha - \beta x_i)^2 \right],$$

and the log-likelihood is

$$\ln L = \sum_{i=1}^{n} \left[ -\frac{1}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} (y_i - \alpha - \beta x_i)^2 \right].$$
Simple Regression in MLE: Estimating $\alpha$

Let $Q = \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$.

$$\frac{\partial Q}{\partial \alpha} = \sum_{i=1}^{n} 2(y_i - \alpha - \beta x_i)(-1) = 0$$

$$\sum_{i=1}^{n} y_i = n\hat{\alpha} + \hat{\beta} \sum_{i=1}^{n} x_i$$

$$\hat{\alpha} = \bar{y} - \hat{\beta} \bar{x}$$
Simple Regression in MLE: Estimating $\beta$

Let $Q = \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$.

$$\frac{\partial Q}{\partial \beta} = \sum_{i=1}^{n} 2(y_i - \alpha - \beta x_i)(-x_i) = 0$$

$$\sum_{i=1}^{n} y_i x_i = \hat{\alpha} \sum_{i=1}^{n} x_i + \hat{\beta} \sum_{i=1}^{n} x_i^2$$

Now substitute in for $\hat{\alpha}$.
Simple Regression in MLE: Estimating $\beta$

$$
\sum_{i=1}^{n} y_i x_i = \hat{\alpha} \sum_{i=1}^{n} x_i + \hat{\beta} \sum_{i=1}^{n} x_i^2
$$

$$
\sum_{i=1}^{n} y_i x_i = (\bar{y} - \hat{\beta} \bar{x}) n \bar{x} + \hat{\beta} \sum_{i=1}^{n} x_i^2
$$

$$
\sum_{i=1}^{n} y_i x_i - n \bar{x} \bar{y} = \hat{\beta} \left( \sum_{i=1}^{n} x_i^2 - n \bar{x}^2 \right)
$$

$$
\hat{\beta} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}
$$
Why We Like ML Estimators

The major properties of MLEs are large sample or asymptotic.

1. They are consistent, \( \text{plim} \left( \hat{\theta} \right) = \theta \).
2. They are asymptotically normal.
3. They are asymptotically efficient.
4. They are invariant to transformation: if \( \hat{\theta} \) is the MLE of \( \theta \) and \( g(\theta) \) is a continuous function of \( \theta \), then \( g(\hat{\theta}) \) is the MLE of \( g(\theta) \).
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4. They are invariant to transformation: if $\hat{\theta}$ is the MLE of $\theta$ and $g(\theta)$ is a continuous function of $\theta$, then $g(\hat{\theta})$ is the MLE of $g(\theta)$. 
Where Do the Standard Errors Come From?

The estimated variation of $\hat{\theta}$ is given by $-H^{-1}$, where $H$ is the matrix of second derivatives (better known as the Hessian),

$$H = \frac{\partial^2 \ln L(\hat{\theta}; X)}{\partial \hat{\theta} \partial \hat{\theta}'} = \frac{\partial^2 \ln \ell(\hat{\theta}; x_i)}{\partial \hat{\theta} \partial \hat{\theta}'} + \cdots + \frac{\partial^2 \ln \ell(\hat{\theta}; x_N)}{\partial \hat{\theta} \partial \hat{\theta}'}$$

The estimated standard errors are the square roots of the diagonal of $-H^{-1}$. 
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$$

The estimated standard errors are the square roots of the diagonal of $-\mathbf{H}^{-1}$. 
We stated earlier that ML estimates are asymptotically normal,

\[ \sqrt{n}(\hat{\theta} - \theta) \overset{d}{\to} N[0, \text{Var}(\hat{\theta})]. \]
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\[ \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N[0, \text{Var}(\hat{\theta})]. \]

If \( g(\theta) \) is some function of \( \theta \), then it can be shown that

\[ \text{Var}(g(\hat{\theta})) = (Dg(\hat{\theta})) \text{Var}(\hat{\theta})(Dg(\hat{\theta}))', \]

where \( D = \frac{\partial}{\partial \theta}. \)
How We Get SEs — Step 2

Let $g(\hat{\theta}) = \frac{\partial \ln L(\hat{\theta}; x)}{\partial \hat{\theta}}$, which is also known as the *score function*. 
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Let \( g(\hat{\theta}) = \frac{\partial \ln L(\hat{\theta}; x)}{\partial \hat{\theta}} \), which is also known as the *score function*.

Therefore,

\[
\text{Var}(g(\hat{\theta})) = (Dg(\hat{\theta})) \text{Var}(\hat{\theta})(Dg(\hat{\theta}))'
\]

\[
= \left( \frac{\partial^2 \ln L(\hat{\theta}; x)}{\partial \hat{\theta} \partial \hat{\theta}'} \right) \text{Var}(\hat{\theta}) \left( \frac{\partial^2 \ln L(\hat{\theta}; x)}{\partial \hat{\theta} \partial \hat{\theta}'} \right)'
\]

\[
= H \text{Var}(\hat{\theta})H
\]
How We Get SEs — Step 2

Let $g(\hat{\theta}) = \frac{\partial \ln L(\hat{\theta}; x)}{\partial \hat{\theta}}$, which is also known as the score function.

Therefore,

$$\text{Var}(g(\hat{\theta})) = (Dg(\hat{\theta})) \text{Var}(\hat{\theta})(Dg(\hat{\theta}))'$$

$$= \left( \frac{\partial^2 \ln L(\hat{\theta}; x)}{\partial \hat{\theta} \partial \hat{\theta}'} \right) \text{Var}(\hat{\theta}) \left( \frac{\partial^2 \ln L(\hat{\theta}; x)}{\partial \hat{\theta} \partial \hat{\theta}'} \right)'$$

$$= H \text{Var}(\hat{\theta})H$$

Thus,

$$\text{Var}(\hat{\theta}) = H^{-1} \text{Var}(g(\hat{\theta}))H^{-1}.$$
How We Get SEs — Step 3

What is $\text{Var}(g(\hat{\theta}))$?
How We Get SEs — Step 3

What is $\text{Var}(g(\hat{\theta}))$?

Since the expected value of the score function is 0, the variance of the score function is the score function squared,

$$
\text{Var}(g(\hat{\theta})) = E[g(\hat{\theta})^2] = E \left[ \left( \frac{\partial \ln L(\hat{\theta}; x)}{\partial \hat{\theta}} \right)^2 \right]
$$

$$
= - E \left[ \frac{\partial^2 \ln L(\hat{\theta}; x)}{\partial \hat{\theta} \partial \hat{\theta}'} \right]
$$

$$
= - H
$$
How We Get SEs — Last Step

Since $\text{Var}(g(\hat{\theta})) = -H$,

$$
\text{Var}(\hat{\theta}) = H^{-1} \text{Var}(g(\hat{\theta})) H^{-1} \\
= H^{-1} (-H) H^{-1} \\
= -H^{-1}
$$
How We Get SEs — Last Step

Since \( \text{Var}(g(\hat{\theta})) = -H \),

\[
\text{Var}(\hat{\theta}) = H^{-1} \text{Var}(g(\hat{\theta})) H^{-1} = H^{-1} (-H) H^{-1} = -H^{-1}
\]

Other names for this result:

- Inverse of the Fisher information matrix
- Cramér-Rao Lower Bound
Why $R$?
Why \( R \)?

- it has no limitations
Why *R*?

- it has no limitations
- the calculations are correct
Why \( R \)?

- it has no limitations
- the calculations are correct
- it’s free!
Enough?

How about a coffee break?