A Model of Limited Foresight*

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Abstract

The paper models an individual who may not foresee all relevant aspects of an uncertain environment. The model is axiomatic and provides a novel choice-theoretic characterization of the subalgebra of foreseen events. It is proved that all recursive, consequentialist models imply perfect foresight and thus cannot accommodate unforeseen contingencies. In particular, the model is observationally distinct from recursive models of ambiguity. The process of learning implied by dynamic behavior generalizes the Bayesian model and permits the subalgebra of foreseen events to expand over time.

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1 Introduction

1.1 Objectives

Contingencies arise that were not foreseen at earlier dates. Individuals adapt their strategies and behavior often reflects the awareness that other unanticipated changes may yet occur. How does one model such behavior?

The standard approach to dynamic choice postulates an idealized individual who comprehends fully the uncertainty describing her environment. She foresees every contingency that may eventuate and knows the outcomes induced by every action in every state of the world. Plans reflect this knowledge and are consistently implemented over time. In effect, dynamic behavior is reduced to the static choice of an optimal strategy.

This paper develops an axiomatic model of dynamic choice in which the individual may not foresee all relevant aspects of an uncertain environment. Two desiderata characterize the model.

First, the individual is self-aware and knows that her perception of the environment may be incomplete. The paper provides a choice-theoretic characterization of the collection of foreseen events and shows that awareness induces a nonsingleton set of beliefs over this collection. The multiplicity of priors reflects a preference for robustness or hedging against unanticipated events.

Second, the paper models a forward-looking individual who plans ahead but also adapts to unforeseen contingencies. As time unfolds, her perception of the environment improves and the individual revises her strategy. A novel axiom, *Weak Dynamic Consistency*, assumes that adaptations arise only when unanticipated events alter the individual’s perception. The implied process of learning generalizes the Bayesian model and permits the collection of foreseen events to expand over time.

The key to developing the model comes from answering the question: *At any point in time, what behavior would reveal the collection of events foreseen by the individual?* In atemporal settings, Epstein, Marinacci, and Seo [4] and Gilboa and Schmeidler [10] show that models of limited foresight are observationally equivalent to ambiguity averse behavior. Since ambiguity aversion is conceptually distinct from limited foresight, the atemporal setting provides no behaviorally meaningful way to define unforeseen events.
This paper provides a choice-theoretic characterization when outcomes unfold over time. The characterization is illustrated in the following example which exhibits the static and dynamic implications of limited foresight and shows how the temporal domain of choice permits the behavioral separation of limited foresight from existing models of uncertainty.

1.2 Motivating Examples

1.2.1 Static Behavior

Consider an individual who chooses between two actions \( L \) and \( R \) whose consequences unfold over a four-period horizon. There is a binary shock in each of the first three periods that may affect payoffs. In Figure 1, the relevant uncertainty is represented diagrammatically as an event tree, where each branch corresponds to a possible state of the world and shocks are depicted as the outcomes of a coin toss. Thus, the uppermost branch corresponds to the history of three successive heads \( \{H_1H_2H_3\} \). The numbers inside the boxes denote state-contingent payoffs. For example, the choice of action \( L \) results in a payoff of $17 in period \( t = 2 \) if the event \( \{H_1H_2\} \) obtains. The depicted representation of the actions \( L \) and \( R \) as mappings from states of the world into payoffs is interpreted as a description of objective reality. The individual’s subjective perception is derived from her ranking of the actions \( L \) and \( R \). For simplicity, consider an individual who is risk-neutral and does not discount the future.\(^1\) Her objective is to maximize cumulative wealth.

The individual who foresees all possible states of the world and knows the corresponding outcomes deduces correctly that both actions \( L \) and \( R \) are effectively certain: the cumulative payoff of either action is $10 in every state of the world. Irrespective of her beliefs, she is then indifferent between \( L \) and \( R \) and the action which pays $10 at the time of choice:

\[
L \sim_0 10 \text{ and } R \sim_0 10 \tag{1.1}
\]

To understand the implications of limited foresight, consider the rankings:

\[
L \prec_0 10 \text{ and } R \sim_0 10 \tag{1.2}
\]

\(^1\)The assumption that the individual does not discount the future is relaxed in the formal model. For ease of exposition, risk neutrality is maintained throughout. It should be viewed as a normalization that can be justified in the familiar way by adopting an Anscombe and Aumann [2] formulation.
How can one interpret this behavior? The effectively certain action \( R \) entails a ten-dollar bet on the event \( H_1 \) paying at period \( t = 2 \) and an analogous bet on the event \( T_1 \) paying at period \( t = 3 \). The indifference \( R \sim_0 10 \) reveals that the individual understands the immediate contingencies \( H_1 \) and \( T_1 \), and in particular, that the two bets offset one another. If similar indifference obtains for all other effectively certain actions involving bets on \( H_1 \) and \( T_1 \), the paper concludes that these events are \textit{subjectively foreseen}.

In contrast, the outcomes of action \( L \) depend on contingencies that resolve in the more distant future. If these contingencies are unforeseen, the individual cannot deduce that distant payoffs offset the more imminent, short-run uncertainty she faces. Thus, she requires a premium as captured by the preference \( L \prec_0 10 \).

The latter ranking cannot be attributed to uncertainty about likelihoods when all events are foreseen. If the individual foresees the possible contingencies and knows the corresponding outcomes, she necessarily exhibits the rankings in (1.1). Since these rankings hold independently of beliefs, their negation reveals limited foresight.

The possibility to test the individual’s knowledge of the environment \textit{independently of her beliefs} is specific to the temporal domain of choice and the existence of nontrivial effectively certain actions. If all uncertainty resolves in a single period, the only effectively certain actions are constant.
acts whose ranking reveals no information about the perception of events. Conversely, any nontrivial act is uncertain and its evaluation requires an assessment of beliefs. Thus, the atemporal domain is too ‘small’ to separate ambiguity aversion from limited foresight. This conclusion is confirmed by Epstein, Marinacci, and Seo [4], Ghirardato [8], Gilboa and Schmeidler [10] and Mukerji [15].

1.2.2 Dynamic Behavior

To illustrate the implications of limited foresight for dynamic behavior, consider the simple example of an individual whose foresight at time $t = 0$ is limited but who understands the environment perfectly at time $t = 1$. Denote her conditional preferences in the latter period by $\succeq_{1,H_1}$ and $\succeq_{1,T_1}$, as the event $H_1$, or respectively $T_1$, obtains. The choices below contrast the individual’s posterior and prior valuations of the actions $L$ and $R$:

$$L \sim_{1,H_1} 10, \quad L \sim_{1,T_1} 10 \quad \text{and} \quad L \prec_0 10$$  \hfill (1.3)

$$R \sim_{1,H_1} 10, \quad R \sim_{1,T_1} 10 \quad \text{and} \quad R \sim_0 10$$  \hfill (1.4)

The rankings in (1.3) and (1.4) reveal two important characteristics of dynamic choice under limited foresight.

The premium required for distant, poorly foreseen bets disappears as time unfolds and the perception of the individual improves. The corresponding rankings (1.3) imply a violation of dynamic consistency that is precluded by the standard approach to dynamic choice. This violation arises as the individual learns aspects of the environment she did not anticipate and could not take into account ex ante.

In contrast, the indifference $R \sim_0 10$ in (1.4) reveals that the individual understands the immediate contingencies $H_1$ and $T_1$. Then, her conditional preferences indicate that she evaluates the action consistently over time.

The example illustrates the approach of this paper to modeling coherent dynamic behavior when some contingencies are unforeseen: The individual is forward-looking and revises her plans only when unanticipated circumstances contradict her perception.
1.2.3 Foresight and Dynamic Consistency

In models of dynamic choice, intertemporal consistency is often motivated by perfect foresight: the individual anticipates all future contingencies and plans ahead. The paper proves that foresight is in fact necessary: *dynamically consistent, consequentialist and state-independent models of behavior imply perfect foresight*, where the latter is defined by the ranking of effectively certain actions.

The result has two important implications. First, any *recursive* model of dynamic behavior precludes the ex ante rankings in (1.2) that motivate this paper. Importantly, the premium on $R$ cannot be interpreted as preference for early resolution of uncertainty studied by Kreps and Porteus [14] and Epstein and Zin [6]. The recursive, intertemporal models of ambiguity in Epstein and Schneider [5] and Klibanoff, Marinacci, and Mukerji [11] similarly preclude such a premium, demonstrating that in a temporal domain of choice limited foresight and ambiguity aversion have distinct testable implications.

Second, a corollary of the result permits an alternative and equivalent characterization of the collection of foreseen events. In example (1.2), the contingencies $H_1$ and $T_1$ foreseen by the individual are revealed by the *static* ranking of effectively certain actions. The paper shows how the *intertemporal consistency* of behavior provides an equivalent characterization. To gain some insight, consider the rankings in (1.3) and (1.4). The violation of dynamic consistency in (1.3) suggests a surprise, or equivalently, that some of the more distant consequences of the action $L$ are not fully anticipated in period $t = 0$. Conversely, the intertemporally consistent ranking of $R$ reveals that the individual understands its constituent bets or equivalently that the immediate contingencies $H_1$ and $T_1$ are *foreseen*. The implied characterization confirms a conjecture by Kreps [13, p.278] that intertemporal choice may reveal the collection of foreseen events and provide a foundation for separating limited foresight from existing models of uncertainty.
2 Static Model

2.1 Domain

The objective environment is described by a state space Ω and a finitely generated filtration $\mathcal{F} := \{\mathcal{F}_t\}$ where time varies over a finite horizon $T = \{0, 1, ..., T\}$. An action taken by the individual induces a real-valued, $\{\mathcal{F}_t\}$-adapted process of outcomes lying in some compact interval $M$. Call any such process an act and denote generic acts by $h, h'$. The set of all acts $\mathcal{H}$ is a mixture space under the obvious operation.

An important part of understanding the model lies in the interpretation of the domain of choice, and specifically, in the implicit distinction between acts and actions. Physical actions, such as the purchase of a dividend-paying stock, comprise the individual’s domain of choice. Acts are a mathematical construct used to model the relevant uncertainty: each act maps states of the world into outcomes. The paper assumes that the modeler observes the choice of action and knows the corresponding acts. Under the assumption that each act is induced by a unique action, the observable choice over actions induces a unique preference over acts. The latter is adopted as a primitive of the model. The objective is to infer the individual’s perception of the environment as a component of the representation.

To proceed, let $\succeq$ denote the preference ordering over the set of acts $\mathcal{H}$. For any act $h$, $\mathcal{F}(h)$ denotes the smallest filtration with respect to which the act $h$ is adapted. Conversely, for any subfiltration $\mathcal{G}$ of $\mathcal{F}$, $\mathcal{H}_\mathcal{G}$ denotes the subset of all $\mathcal{G}$-adapted acts. For any $t$, $\Pi_{\mathcal{G}_t}$ is the partition generating the algebra $\mathcal{G}_t$ and, for any $\omega$, $\mathcal{G}_t(\omega)$ is the atom in $\Pi_{\mathcal{G}_t}$ containing $\omega$. Since $\mathcal{F}_T$ is finite, the latter is well-defined. It is assumed that $\mathcal{F}_T = \mathcal{F}_{T-1}$. Thus, the individual lives for another period after all relevant uncertainty is resolved. The assumption implies that the subset of effectively certain acts is rich. In particular, it generates the filtration $\mathcal{F}$.

Generic outcomes in $M$ are denoted by $x, x', y$. The deterministic act paying $x$ in each period and each state of the world is denoted by $x$. For any act $h$ and state $\omega$, $h(\omega)$ is the deterministic act which pays $h_t(\omega)$ in period $t$.

\footnote{The assumption is redundant in an infinite-horizon model. See the next section for further discussion.}
2.2 Definition of Foreseen Events

The section introduces the behavioral definition of foreseen events. The explicit, choice-theoretic characterization ensures verifiability given suitable choice data and thus the potential empirical relevance of unforeseen contingencies. The main preliminary step identifies a class of actions whose ranking does not require an assessment of likelihoods, but which is sufficiently rich to reveal the individual’s knowledge of the environment.

Building on the introductory example, define an action $h$ to be **effectively certain** if

$$h(\omega) \sim h(\omega') \text{ for all } \omega, \omega' \in \Omega.$$  

Theorem 6 shows that any effectively certain action $h$ is necessarily indifferent to the deterministic action $h(\omega)$ whenever preference can be represented by a recursive utility function. The motivating example suggests that such indifference is intuitive *only if* the individual has perfect knowledge of the relevant environment. Consequently, and in contrast to the standard model, the paper does not impose indifference for all effectively certain actions *a priori*. Instead, it takes the subset of actions for which indifference obtains as *indicative* of the collection of foreseen events.

An effectively certain action $h$ is **subjectively certain** if for all effectively certain, $\mathcal{F}(h)$-adapted actions $h'$:

$$h' \sim h'(\omega) \text{ for all } \omega \in \Omega.$$  

Two complementary requirements comprise the definition of subjective certainty. First, any subjectively certain action $h$ must be indifferent to $h(\omega)$ for every $\omega \in \Omega$. This requirement is satisfied by the action $R$ in the introductory example (1.2). There, the indifference $R \sim 10$ reveals that the individual *foresees* the contingencies $H_1$ and $T_1$. This interpretation is strengthened by the second requirement of subjective certainty. It posits that the same indifference obtains for all other $\mathcal{F}(h)$-adapted, effectively certain actions. To take a simple example, replace the $\$10$ outcomes of action $R$ with $\$15$. By construction, the new action, $h'$, is effectively certain and, moreover, entails bets on the same events $H_1$ and $T_1$. The subjective certainty of $h$ then requires that $h'$ is indifferent to the sure payment of $\$15$. In effect, subjective certainty is a property of the *events* that constitute and describe
the uncertainty pertaining to the evaluation of the action \( h \). In particular, the property is robust to changes in the action’s outcomes. The discussion motivates the behavioral definition of foreseen events.

**Definition 1** An event is **foreseen** if it belongs to the filtration, \( G \), induced by the subset of subjectively certain acts. An act is **foreseen** if it is \( G \)-adapted.

The scope and applicability of this definition merit some discussion. As a behavioral test of unforeseen contingencies, subjective certainty is powerful *only* in a multi-period domain of choice. If all uncertainty resolves in a single period, the ranking \( h \sim h(\omega) \) may be alternatively interpreted as an instance of state-dependent preferences. This interpretation is *not* valid in a temporal setting because subjective certainty is *no longer* implied by state-independence. The latter is evident from the ranking \( L < 10 \) in (1.2) and the fact that preferences in the introductory example are risk-neutral and *a fortiori* state-independent.

The temporal domain is important in that it permits the construction of nontrivial effectively certain actions. To take a simple example, consider the construction of the action \( R \): for a given event \( A \), there is a \$10 bet on \( A \) that pays off in period \( k \), and an analogous bet on \( A^c \) that pays off in period \( k' \neq k \). The noted requirement is that the event \( A \) belongs to at least two algebras \( \mathcal{F}_k \neq \mathcal{F}_{k'} \) in the objective event tree \( \{\mathcal{F}_t\} \). In a finite-horizon model, the latter holds if and only if \( A \in \mathcal{F}_{T-1} \). Consequently, the proposed distinction between ‘foreseen’ and ‘unforeseen’ is empirically relevant only for events in \( \mathcal{F}_{T-1} \). In particular, the distinction becomes moot in the extreme case when \( \mathcal{F}_0 = \mathcal{F}_{T-1} \). The latter case is isomorphic to an atemporal model since all certainty resolves in a single period. For expositional ease, the paper assumes that \( \mathcal{F}_T = \mathcal{F}_{T-1} \), or equivalently, that the distinction is relevant for all events in the objective environment.

The definition proposes a test of unforeseen contingencies that is independent of the individual’s assessment of likelihoods. This implies that the test is necessarily minimal in terms of its predictive power: the subset of effectively certain actions is the *smallest* subset whose ranking is *sufficient* to identify the collection of foreseen events. The empirical content of limited foresight, however, is not limited to the specification of foreseen events. How does a self-aware individual perceive and evaluate actions whose outcomes depend
on unforeseen states of the world? How do we model learning in response to objective information? The second question is especially pertinent since limited foresight necessitates a weakening of dynamic consistency and thus a generalization of Bayesian updating. An answer to these questions requires that the definition is incorporated in a complete model of dynamic choice. The rest of the paper proceeds axiomatically to develop and characterize such a model.

2.3 Axioms

The individual’s choices among physical actions induce a preference ordering on the set of objective acts $\mathcal{H}$. This section adopts a set of axioms on the induced preference $\succeq$. Some of the axioms have a standard interpretation if the individual has perfect knowledge of the environment. In this case, her subjective perception and the primitive objective environment coincide. If perception is coarse, however, the axioms make implicit assumptions about how perception differs from, and approximates, the objective world. The first two axioms fall in this category and their content is reinterpreted accordingly.

**Basic** The preference $\succeq$ is complete and transitive, mixture-continuous and monotone.

Monotonicity requires that an action $h$ is preferred to $h'$ whenever the outcomes of the former exceed the outcomes of $h'$ in every period and every state of the world. The axiom remains compelling even if the individual has an incomplete perception of the objective environment. To take an extreme example, imagine an individual whose perception of the world is trivial: she knows only that ‘something’ may happen. For any given action, she foresees the worst possible outcome and ranks actions accordingly. Then, her behavior is (weakly) monotone. Intuitively, the individual may have an incomplete perception of the environment but need not be delusional and prefer an action $h'$ to another action that dominates it.

**Convexity** For all $h, h' \in \mathcal{H}$, $h \sim h'$ implies $\alpha h + (1 - \alpha)h' \succeq h$.

To understand Convexity, it is useful to imagine a hypothetical, auxiliary step in which the individual is asked to compare the subjective mixture of $h$
and \( h' \) to either action. That is, the individual can mix the outcomes of \( h \) and \( h' \) as she perceives them. The subjective mixture ‘smooths’ outcomes across states foreseen by the individual. Aware that her perception may be incomplete, the individual prefers the mixture. The latter hedges her exposure to contingencies that she fears might be only a coarse approximation to the world. Convexity requires that the objective mixture \( \alpha h + (1 - \alpha)h' \) be preferred to its subjective counterpart. The former smooths the uncertainty within as well as across any of the foreseen events.

**Strong Certainty Independence** For all acts \( h, h' \in \mathcal{H} \) and effectively certain, foreseen acts \( g \):

\[
h \succeq h' \text{ if and only if } \alpha h + (1 - \alpha)g \succeq \alpha h' + (1 - \alpha)g.
\]

To understand Strong Certainty Independence, imagine the mixture of the action \( L \) and the subjectively certain action \( R \) in examples (1.1) and (1.2). Since \( R \) is foreseen, it is constant within the events \( H_1 \) and \( T_1 \) foreseen by the individual. As such, it cannot hedge the poorly understood uncertainty within these events. Since the action is effectively certain, it also provides no hedging across the collection of foreseen events. The conjunction of these arguments motivates Strong Certainty Independence.

To illustrate the next axiom, consider the foreseen action \( R \) in the introductory example and the corresponding ranking \( R \sim 10 \). The action \( R \) entails a ten-dollar bet on the event \( H_1 \) paying in period \( t = 2 \) and an analogous bet on the event \( T_1 \) paying in period \( t = 3 \). Suppose one were to delay the payment of these bets by one period. The new action entails bets on the same events \( H_1 \) and \( T_1 \) but pays in periods \( t = 3 \) and \( t = 4 \), respectively. Since the events are foreseen, Stationarity requires that the ranking remains the same.

**Stationarity** For all acts \( h, h' \in \mathcal{H} \) and for all outcomes \( x \in M \),

\[
(h_0, \ldots, h_{T-1}, x) \succeq (h'_0, \ldots, h'_{T-1}, x) \text{ if and only if } (x, h_0, \ldots, h_{T-1}) \succeq (x, h'_0, \ldots, h'_{T-1}),
\]

whenever the acts on the left (right) are foreseen.

The set of nodes \( \bigcup_t \Pi_t \mathcal{G}_t \) in the filtration of foreseen events correspond to states of the world as perceived by the individual. The next axiom is
a subjective analogue of the standard monotonicity or state-independence assumption applied to these subjective states.

**Subjective Monotonicity** For all acts $h, h' \in \mathcal{H}$ and for all outcomes $x \in M$,

$$hAx \succeq h'Ax \text{ for all } A \in \bigcup_i \Pi_{G_i} \text{ implies } h \succeq h'.$$

Consider an action $h$ whose continuation act at some node $\mathcal{F}_t(\omega)$ is non-constant: $h_\tau(\omega') \neq h_\tau(\omega'')$ for some $\tau > t$ and $\omega', \omega'' \in \mathcal{F}_t(\omega)$. Say that $h'$ *simplifies* $h$ if $h'$ has a constant continuation at $\mathcal{F}_t(\omega)$ and equals $h$ elsewhere. By construction, $h'$ depends on events that are strictly closer in time. The next axiom requires that $h'$ is foreseen whenever $h$ is. Thus, events closer in time are easier to foresee.

**Sequentiality** If $g'$ simplifies a foreseen act $g$, then $g'$ is foreseen.

Define nullity in the usual way: the event $A \in \mathcal{F}_T$ is $\succeq$-null if $h(\omega) = h'(\omega)$ for all $\omega \in A^c$ implies $h \sim h'$.

**Nonnullity** Every nonempty foreseen event is nonnull.

### 2.4 Representation

#### 2.4.1 Subjective Filtration

The section introduces the class of filtrations used to model the individual’s perception of the objective environment $(\Omega, \{\mathcal{F}_t\})$.

**Definition 2** A subfiltration $\{\mathcal{G}_t\}$ of $\{\mathcal{F}_t\}$ is **sequentially connected** if

$$\Pi_{\mathcal{G}_t} \setminus \Pi_{\mathcal{F}_t} \subset \Pi_{\mathcal{G}_{t+1}} \text{ for all } t < T.$$
$A_1 \cup A_2 \in \Pi_{G_{t+1}}$. The requirement captures the intuition that events more distant in time are more difficult to foresee.

It is not difficult to see that any sequentially connected filtration $\{G_t\}$ is fully determined by the algebra $G_T$:

$$G_t = \mathcal{F}_t \cap G_T \text{ for all } t \in T. \quad (2.1)$$

Define an algebra to be sequentially connected, if it induces a sequentially connected filtration via (2.1). The rest of the paper uses $G$ interchangeably to denote the filtration and the algebra which generates it.

Sequentially connected filtrations include a number of common and intuitive specifications.

**Example 1** (Fixed Horizon) The individual foresees all events up to some period $k$:

$$G_t = \mathcal{F}_t \text{ for all } t \leq k \text{ and } G_t = \mathcal{F}_k \text{ for } t > k.$$ 

More generally, the individual may not foresee the contingencies describing an unlikely event $A$, but have a better understanding of its complement. Her depth of foresight is then a random variable and the corresponding sequentially connected algebra can be modeled as a stopping time.

**Example 2** (Random Horizon) The individual foresees all events up to a stopping time $\tau$, where

$$\tau : \Omega \rightarrow T \text{ such that } \{\omega : \tau(\omega) = k\} \in \mathcal{F}_k \text{ for all } k \in T.$$

The filtration $\{G_t\}$ induced by the stopping time $\tau$

$$G_t := \{A \in \mathcal{F}_t : A \cap \{\omega : \tau(\omega) = k\} \in \mathcal{F}_k \text{ for all } k \in T\} \text{ for } t \in T$$

is sequentially connected.\(^3\)

Sequentially connected filtrations arise as the outcome of a satisficing procedure for simplifying decision trees proposed by Gabaix and Laibson [7] in a setting of objective uncertainty.

\(^3\)Appendix 5.6 shows that sequentially connected filtrations inherit the lattice structure of stopping times. Specifically, the supremum (infimum) of sequentially connected filtrations is sequentially connected.
Example 3 (Satisficing) The individual ignores branches of the decision tree whose probability is lower than some threshold \( \alpha \in [0, 1] \).

The Gabaix and Laibson [7] procedure permits a parsimonious parametrization of sequentially connected filtrations via the threshold parameter \( \alpha \).

The class of sequentially connected filtrations excludes the following subfiltration:

\[
\mathcal{G} = \{ \mathcal{F}_0, \mathcal{F}_0, \ldots, \mathcal{F}_0, \mathcal{F}_T \}
\]

In this example, the individual foresees all possible contingencies (\( \mathcal{G}_T = \mathcal{F}_T \)) but she ‘delays’ the resolution of uncertainty. That is, she believes erroneously that all information is revealed in the last period. Since the filtration does not capture a coarse perception of the environment, it is precluded by Definition 2.

2.4.2 Subjective Acts

In Savage’s [16] theory of subjective probability, the individual behaves as if she contemplates all possible states of the world and anticipates the outcomes that any given physical action might induce. That is, it is as though the individual herself perceives physical actions as the acts describing the relevant, objective uncertainty.

This implication of Savage’s theory is no longer valid when the subjective filtration \( \mathcal{G} \) is coarse. More precisely, the individual’s perception of actions might differ from, and only approximate, the true objective acts. The suggested difference can be represented diagrammatically:

\[
\begin{array}{ccc}
\text{Physical actions} & \leftrightarrow & \text{Objective acts} \\
\Phi & \downarrow & \Phi \\
\text{Subjective acts}
\end{array}
\]

Appendix 5.6 provides a detailed translation of the Gabaix-Laibson definition into the setting of this paper.
The diagram relates the individual’s subjective perception of uncertainty to the observable primitives of the model and their interpretation as given in Section 2.1. Specifically, it is assumed that the modeler observes the individual’s choices among physical actions and maps each physical action into a unique act in the set $\mathcal{H}$. The latter is interpreted as an objective description of the relevant environment in the sense that it can be constructed by the modeler or an outside observer. The function $\Phi$ admits the possibility that the individual’s perception of the environment is coarse, and thus, takes each physical action into a possibly different, subjective act. Finally, the mapping $\Phi$ makes the graph commute.

The diagram makes it clear that even though physical actions are not explicitly modeled, the individual’s perception of actions can be represented directly by means of a mapping $\Phi$ from objective into subjective acts. From this perspective, Savage’s axioms can be viewed as implying a representation in which $\Phi$ is the identity mapping from the set of objective acts $\mathcal{H}$ into itself. This reflects the perfect foresight of the Savage individual. The next definition introduces an appropriate generalization that permits the modeling of unforeseen contingencies.

**Definition 3** A continuous, monotone and concave function $\Phi : \mathcal{H} \rightarrow \mathcal{H}_G$ is a $\mathcal{G}$-approximation mapping if:

(i) $\Phi$ is $\mathcal{G}$-additive: $\Phi(ah + g) = a\Phi(h) + g$ for all $h \in \mathcal{H}$ and $g \in \mathcal{H}_G$;

(ii) $\Phi$ is separable: $h_t|A = h'_t|A$ implies $(\Phi h)_t|A = (\Phi h')_t|A$ for all $t$ and all events $A \in \mathcal{G}_t$.

Subjective acts respect the filtration $\mathcal{G}$ of foreseen events. Formally, a $\mathcal{G}$-approximation mapping takes each objective act into a $\mathcal{G}$-adapted subjective act. This property captures the intuition that an individual cannot imagine bets on contingencies she does not foresee. In addition, $\mathcal{G}$-additivity implies that the function $\Phi$ maps each $\mathcal{G}$-adapted objective act into itself. Thus, subjective perception and objective truth coincide for any action that is foreseen. The next example provides a simple, albeit extreme illustration of a $\mathcal{G}$-approximation mapping.
Example 4 (Approximation From Below) For every $h \in \mathcal{H}$, let $\Phi h$ be the lower $\mathcal{G}$-adapted envelope of $h$:

$$(\Phi h)_t|A := \min_{\omega \in A} h_t(\omega), \text{ for every } t \text{ and } A \in \Pi_{\mathcal{G}_t}. \quad (\Phi)$$

Then, $\Phi$ is a $\mathcal{G}$-approximation mapping.

More generally, the subjective act $\Phi h$ is bounded from below (above) by the lower (upper) $\mathcal{G}$-adapted envelope of the objective act $h$. That is,

$$\min_{\omega \in A} h_t(\omega) \leq (\Phi h)_t|A \leq \max_{\omega \in A} h_t(\omega),$$

for every period $t \in \mathcal{T}$ and event $A \in \Pi_{\mathcal{G}_t}$. The proof follows from the next lemma, which provides a convenient way to parametrize the class of $\mathcal{G}$-approximation mappings. The result is a corollary of Gilboa and Schmeidler [9, Theorem 1].

Lemma 1 $\Phi$ is a $\mathcal{G}$-approximation mapping if and only if for every $t$ and every $A \in \Pi_{\mathcal{G}_t}$ there exist a nonempty, closed, convex subset $C_A$ of $\Delta(A, \mathcal{F}_t)$ such that:

$$(\Phi h)_t|A = \min_{p \in C_A} \int_A h_t \, dp. \quad (2.2)$$

2.4.3 Representation Theorem

This section completes the description of the static model of limited foresight.

Definition 4 A preference relation $\succeq$ on $\mathcal{H}$ has a limited foresight representation $(\mathcal{G}, \Phi, C)$ if it admits a utility function of the form:

$$V(h) = \min_{p \in C} \int_{\Omega} \sum_t \beta^t (\Phi h)_t \, dp, \quad (2.3)$$

where $\beta > 0$, $\mathcal{G}$ is the filtration of foreseen events, $\mathcal{G}$ is sequentially connected, $\Phi$ is a $\mathcal{G}$-approximation mapping, and $C$ is a closed, convex subset in the interior of the simplex $\Delta(\Omega, \mathcal{G}_T)$.

The filtration $\mathcal{G}$ and the mapping $\Phi$ define the individual’s subjective model of the environment - the contingencies she foresees and her perception of actions. The nonsingleton set of priors $C$ reflects her awareness that this model may be incomplete.
Theorem 2 A preference $\succeq$ satisfies Basic, Convexity, Strong Certainty Independence, Stationarity, Sequentiality and Nonnullity if and only if it has a limited foresight representation $(\mathcal{G}, \Phi, C)$. Moreover, the latter is unique.

2.4.4 Discussion

To apply the model, it is first necessary to specify the components of the representation: a sequentially connected filtration $\mathcal{G}$, a mapping $\Phi$ and a set of priors $\mathcal{C}$ defined on the algebra $\mathcal{G}_T$. Preference is then induced via the utility function in (2.3). A requirement of the representation is that $\mathcal{G}$ is the filtration of foreseen events for the induced preference. The requirement is not satisfied for any choice of utility components. To emphasize its significance, this section gives an example of a triple $(\mathcal{G}', \Phi', C')$ for which $\mathcal{G}'$ is strictly coarser than the filtration of foreseen events. Sufficient conditions on the mapping $\Phi'$ are then provided that guarantee the required property and hence that $(\mathcal{G}', \Phi', C')$ constitutes a limited foresight representation.

To describe the example, consider the triple $(\mathcal{G}_0, \Phi^0, C^0)$, where $\mathcal{G}_0$ is the trivial filtration, $C^0$ is the degenerate measure on $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and where the mapping $\Phi^0$ is defined as follows:

$$(\Phi^0 h)_t(\omega) := \int h_t d\omega, \text{ for every } \omega \in \Omega \text{ and } t \in T, \quad (2.4)$$

for some measure $\mu$ on $\mathcal{F}_T$. The trivial filtration suggests the interpretation of zero foresight: the individual knows that ‘something may happen’ but is unable to specify any finer contingencies. Accordingly, the mapping $\Phi^0$ takes each objective act into a deterministic subjective act.

Next, consider the requirements of a limited foresight representation. It is easy to see that the trivial filtration is sequentially connected and that, by Lemma 1, the function $\Phi'$ is a $\mathcal{G}'$-approximation mapping. It remains to verify whether $\mathcal{G}'$ is the collection of foreseen events for the induced preference. To do so, consider the utility function in (2.3):

$$V'(h) := \sum_t \beta'(\Phi'h)_t. \quad (2.5)$$

Substituting (2.4) into (2.5) and rearranging, one obtains:

$$V'(h) = \sum_t \beta' \left( \int h_t d\mu \right) = \int \left( \sum_t \beta'h_t \right) d\mu \quad (2.6)$$

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It is evident from the expected-utility representation in (2.6) that all events are foreseen in the sense of Definition 1. Conclude that the triple \((G', \Phi', C')\) is not a limited foresight representation of the induced preference. In fact, the unique such representation \((G, \Phi, C)\) is given by the expected-utility functional in (2.6) where \(G\) is the objective filtration \(\mathcal{F}\), \(\Phi\) is the identity mapping from \(\mathcal{H}\) into itself, and beliefs \(C\) consist of the single prior \(p\).

The example emphasizes the importance of an explicit choice-theoretic definition of foreseen events. For arbitrary triples \((G_0, \Phi_0, C_0)\), the intended separation between ‘foreseen’ and ‘unforeseen’ as captured by the functional component \(G'\) has little or no behavioral content. In contrast, a limited foresight representation \((G, \Phi, C)\) requires that the filtration \(G\) is consistent with the behavioral definition of foreseen events. Thus, the functional component \(G\) translates directly into properties of the observable, and readily interpretable, ranking of effectively certain actions.

The example raises the question if there exist common and easily constructible specifications \((G, \Phi, C)\) which guarantee that \(G\) is the filtration of foreseen events for the induced preference. The next lemma gives sufficient conditions in terms of the mapping \(\Phi\) and shows that the required property holds generically. To state the result, fix a sequentially connected filtration \(G\) and a set of priors \(C\) in \(\Delta(\Omega, \mathcal{G}_T)\). Recall that, by Lemma 1, every \(G\)-approximation mapping \(\Phi\) may be identified with a collection \(\{C_A\}\) where \(C_A\) is a closed, convex subset of \(\Delta(A, \mathcal{F}_t)\) and the event \(A\) varies over cells of the filtration \(G\).

**Lemma 3** If \(C_A\) has nonempty interior in \(\Delta(A, \mathcal{F}_t)\) for every \(A \in \Pi_G\), and every \(t \in T\), then the triple \((G, \{C_A\}, C)\) is a limited foresight representation of the induced preference.

The following specifications of the mapping \(\Phi\) satisfy the sufficient conditions.

**Example 5** (Approximation From Below) For every \(t \in T\), and \(h \in \mathcal{H}\), \((\Phi h)_t\) is the lower \(G_t\)-measurable envelope of \(h_t\). Then \((G, \Phi, C)\) is a limited foresight model.

A notable feature of Example 5 is that the model is fully specified by the filtration \(G\) and the set of priors \(C\). A drawback is the ‘coarseness’ of the
approximation mapping. In terms of the representation derived in Lemma 1, each set $C_A$ in the construction of $\Phi$ equals the entire simplex $\Delta(A)$. To provide a less extreme approximation, one may take the set $C_A$ to be an $\epsilon$-contraction of the simplex around a focal measure $p^*$ in $\Delta(A)$:

$$C_A := \{\epsilon p + (1 - \epsilon)p^* : p \in \Delta(A)\}.$$ 

The corresponding mapping $\Phi_\epsilon$ is determined by the filtration $\mathcal{G}$ and the single parameter $0 < \epsilon < 1$.

**Example 6 (\(\epsilon\)-Contamination)** For every $t \in T$ and every $A \in \Pi_{G_t}$, $C_A$ is an $\epsilon$-contraction of the simplex. Then $(\mathcal{G}, \Phi_\epsilon, \mathcal{C})$ is a limited foresight model.

### 2.4.5 Observational Equivalence in an Atemporal Domain

### 3 Dynamic Model

#### 3.1 Axioms

This section develops the dynamic model of limited foresight. The primitive is an $\mathcal{F}$-adapted process of conditional preferences $\{\succeq_{t,\omega}\}$ where $\succeq_{t,\omega}$ describes the ranking of actions in state $\omega$ and period $t$. It is assumed that each conditional preference $\succeq_{t,\omega}$ admits a limited foresight representation $(\mathcal{G}_{t,\omega}, \Phi_{t,\omega}, C_{t,\omega})$.

The first axiom requires that conditional preferences at $t$ and $\omega$ do not take into account what the actions might yield in any alternative history.

**Consequentialism** For each $t$ and $\omega$ and all acts $h, h'$,

$$h_\tau(\omega') = h'_\tau(\omega') \text{ for all } \tau \geq t \text{ and } \omega' \in F_t(\omega) \implies h \sim_{t,\omega} h'.$$

Consequentialism restricts the scope of limited foresight modeled in this paper. The stated indifference requires that whenever the relevant continuation acts are objectively identical, they are perceived as identical. Intuitively, perception may be incomplete, but not delusional. The individual does not imagine differences if there are none.

The next axiom requires that *tastes* do not depend on the time and state of the world. As in the static model, limited foresight pertains to
the individual’s perception of uncertainty and has no implications for the evaluation of deterministic acts.

**State Independence** For each \( t \) and \( \omega \),

\[
(x_0, \ldots, x_{t-1}, y_t, \ldots, y_T) \succeq_{t,\omega} (x_0, \ldots, x_{t-1}, y'_t, \ldots, y'_T) \text{ if and only if }

(x_0, \ldots, x_{t-1}, y_t, \ldots, y_T) \succeq_0 (x_0, \ldots, x_{t-1}, y'_t, \ldots, y'_T).
\]

To understand the next axiom, consider an environment in which one of three events \( A_1, A_2 \) and \( A_3 \) can be realized in period \( t = 1 \). An individual with perfect foresight evaluates all actions consistently:

\[
h \succeq_{1,A_i} h' \text{ for all } i \text{ implies } h \succeq_0 h'. \tag{3.1}
\]

If she were to learn \( A_1 \cup A_2 \) at some hypothetical intermediate stage \( \tau \), then intertemporal consistency would similarly imply:

\[
h \succeq_{1,A_1} h' \text{ and } h \succeq_{1,A_2} h' \text{ implies } h \succeq_{\tau,A_1\cup A_2} h'. \tag{3.2}
\]

The implications in (3.1) and (3.2) reflect an individual who plans consistently. She knows all possible contingencies and anticipates accurately the future choices she is going to make. This knowledge is incorporated in her behavior in the period \( t = 0 \). It is important to emphasize that the hypothetical stage \( \tau \) and the preferences \( \succeq_{\tau,A_1\cup A_2}, \succeq_{\tau,A_3} \) in (3.2) are *not* part of the formal model. However, they prove useful in interpreting the implications of limited foresight for dynamic behavior.

To continue the example, consider an individual who initially foresees only the events \( A_1 \cup A_2 \) and \( A_3 \). Thinking of the future, she contemplates her behavior conditional on the events she foresees. If \( \succeq^a_{1,A_1\cup A_2} \) and \( \succeq^a_{1,A_3} \) denote these *anticipated preferences*, then:

\[
h \succeq^a_{1,A_1\cup A_2} h' \text{ and } h \succeq^a_{1,A_3} h' \text{ implies } h \succeq_0 h'. \tag{3.3}
\]

As in (3.1) and (3.2), the implication in (3.3) describes an individual who is forward-looking and plans ahead. However, the anticipated preferences reflect prior foresight and may differ from the individual’s *actual* future behavior. Furthermore, the anticipated preferences may differ from the hypothetical rankings \( \succeq_{\tau,A_1\cup A_2} \) and \( \succeq_{\tau,A_3} \) used in (3.2). It turns out that the latter
difference is easier to analyze since the conditioning events, namely, $A_1 \cup A_2$ and $A_3$, are the same in both cases. Specifically, the preference $\succeq^{1, A_1 \cup A_2}_{\tau}$ represents behavior if the individual were to learn the event $A_1 \cup A_2$ and perceive the world as she does at $t = 0$. In contrast, the preference $\succeq^{\tau, A_1 \cup A_2}_{\tau}$ represents behavior if the individual were to learn the event $A_1 \cup A_2$ and perceive the world as she does at $t = 1$.

These perceptions necessarily coincide only when the actions $h$ and $h'$ are foreseen at $t = 0$. Then (3.2) and (3.3) imply:

\[
\begin{align*}
    h \succeq^1_{1, A_i} h' \quad \text{for all } i & \implies h \succeq^{\tau, A_1 \cup A_2}_{\tau} h' \text{ and } h \succeq^{\tau, A_3}_{\tau} h' \\
    & \implies h \succeq^a_{1, A_1 \cup A_2} h' \text{ and } h \succeq^a_{1, A_3} h' \\
    & \implies h \succeq^0_h h'.
\end{align*}
\]

The above implications motivate the next axiom. It formalizes the present approach to modeling sophisticated dynamic behavior when some contingencies are unforeseen: the individual is forward-looking and revises her plans only when unanticipated circumstances contradict her perception. The approach is illustrated in the introductory examples (1.3) and (1.4).

**Weak Dynamic Consistency** For each $t$ and $\omega$ and for all acts $g, g'$ in $\mathcal{H}_{G^t, \omega}$ such that $g^\tau = g'^\tau$ for all $\tau \leq t$,

\[
g \succeq^t_{t+1, \omega} g' \quad \text{for all } \omega' \text{ implies } g \succeq^t_{t, \omega} g'. \tag{3.4}
\]

The stronger axiom, **Dynamic Consistency**, requires that the implication in (3.4) holds for all acts $h, h'$ in $\mathcal{H}$ for which $h^\tau = h'^\tau$ for all $\tau \leq t$.

### 3.2 Representation

The process of learning implied by Consequentialism, State Independence and Weak Dynamic Consistency is derived. Specifically, the section characterizes how perception of the environment $\{G^t, \Phi^t\}$ and beliefs $\{C^t\}$ evolve over time.

The first implication captures a notion of expanding foresight. That is, for every $t$ and $\omega$, the posterior filtration $G^{t+1}_{t+1, \omega}$ refines the prior filtration
$G^t \omega$. To state this formally, let $G^t \omega \cap F_{t+1}(\omega)$ denote the restriction of the prior filtration $G^t \omega$ to the subtree emanating from the event $F_{t+1}(\omega)$.\footnote{For every subset $A$ of $\Omega$, an algebra $\mathcal{G}$ on $\Omega$ induces the algebra $\mathcal{G} \cap A := \{B \cap A : B \in \mathcal{G}\}$ on $A$. A filtration $\{\mathcal{G}_t\}$ induces the filtration $\{\mathcal{G}_t \cap A\} := \{\mathcal{G}_t \cap A\}$ on $A$.} The latter event is realized and is thus known by the individual at period $t + 1$ and state $\omega$.

**Definition 5** A process of filtrations $\{G^t \omega\}$ is refining if $G^{t+1} \omega$ refines $G^t \omega \cap F_{t+1}(\omega)$ for all $t$ and $\omega$.

The next step describes how conditional beliefs, $C^t \omega$, evolve over time. Some preliminary definitions are necessary. For a set of priors $C$ on the objective algebra $F_T$, define the set of Bayesian updates by

$$C_t(\omega) := \{p(\cdot | F_t(\omega)) : p \in C\},$$

and define the set of conditional one-step-ahead measures by

$$C^{t+1}_t(\omega) := \{\text{marg}_{F_{t+1}} p : p \in C_t(\omega)\}.$$

The following definition generalizes the familiar decomposition of a measure in terms of its conditionals and marginals to the decomposition of a set of measures $C$. The requirement is studied in Epstein and Schneider [5], who discuss its role for modeling dynamically consistent behavior when the individual has more than a single prior. Formally, define a set $C$ to be $\{F_t\}$-rectangular if for all $t$ and $\omega$,

$$C_t(\omega) = \{\int_{\Omega} p_{t+1}(\omega') \, dm : p_{t+1}(\omega') \in C_{t+1}(\omega') \text{ for all } \omega' \text{ and } m \in C^{t+1}_t(\omega')\}.$$

The main feature of rectangularity is that the decomposition on the right combines a marginal from $C^{t+1}_t(\omega)$ with any measurable selection of conditionals. This will typically involve ‘foreign’ conditionals. If the set $C$ is a singleton, there are no foreign conditionals and the definition of rectangularity reduces to the standard decomposition of a probability measure.

Turn to the process of updating and focus on the special case when all conditional beliefs $C^t \omega$ are singleton sets. The next definition posits the existence of a ‘shadow’ probability measure $C$ defined on the objective algebra...
\( \mathcal{F}_T \). For every \( t \) and \( \omega \), the conditional belief \( \mathcal{C}^{t,\omega} \) is the Bayesian update of the shadow measure, restricted to the respective collection of foreseen events \( \mathcal{G}^{t,\omega} \). In the general case, \( \mathcal{C} \) is a possibly nonsingleton set of measures and updating proceeds prior-by-prior.

**Definition 6** A process \( \{ \mathcal{C}^{t,\omega}, \mathcal{G}^{t,\omega} \} \) admits a consistent extension if there exists an \( \{ \mathcal{F}_t \} \)-rectangular, closed and convex subset \( \mathcal{C} \) of \( \Delta^\omega(\Omega, \mathcal{F}_T) \) such that
\[
\mathcal{C}^{t,\omega} = \{ \text{marg}_{\mathcal{G}^{t,\omega}} p : p \in \mathcal{C}_t(\omega) \} \text{ for each } t \text{ and } \omega.
\]

Theorem 4 below proves that Consequentialism, State Independence and Weak Dynamic Consistency are necessary and sufficient for the process of learning described by Definitions 5 and 6.

**Theorem 4** A family \( \{ \succeq_{t,\omega} \} \) of limited foresight preferences satisfies Consequentialism, State Independence and Weak Dynamic Consistency if and only if \( \{ \mathcal{G}^{t,\omega} \} \) is refining and \( \{ \mathcal{C}^{t,\omega}, \mathcal{G}^{t,\omega} \} \) admits a consistent extension \( \mathcal{C} \).

The next theorem proves that the consistent extension \( \mathcal{C} \) is unique whenever the individual foresees all one-step-ahead contingencies. That is, for every \( t \) and \( \omega \), \( \mathcal{F}_{t+1}(\omega) \) belongs to the algebra of foreseen events \( \mathcal{G}^{t,\omega} \).

**Theorem 5** If \( \mathcal{F}_{t+1}(\omega) \in \mathcal{G}^{t,\omega} \) for all \( t \) and \( \omega \), the consistent extension \( \mathcal{C} \) provided by Theorem 4 is unique.

### 3.3 Foresight and Dynamic Consistency

Kreps [13, p.278] conjectured that dynamic behavior may reveal the collection of foreseen events and provide a foundation for separating limited foresight from existing models of uncertainty. The next theorem establishes a close connection between the intertemporal consistency of behavior and the static ranking of effectively certain actions. As a corollary, the section shows that dynamic behavior provides an equivalent and alternative characterization of the process of subjective filtrations \( \{ \mathcal{G}^{t,\omega} \} \).

\(^6\)Recall that \( \mathcal{G}^{t,\omega} \) denotes both the sequentially connected filtration at \( t, \omega \) and the algebra which generates it via (2.1).
Theorem 6 If a family of preferences $\{\succeq_{t,\omega}\}$ satisfies Consequentialism, State Independence and Dynamic Consistency, then all events are foreseen in period $t = 0$. That is, for all acts $h$:

$$h(\omega) \sim_0 h(\omega') \text{ for all } \omega, \omega' \in \Omega \implies h \sim_0 h(\omega).$$  \hspace{1cm} (3.5)

It is intuitive that any meaningful definition of foreseen events must be related to the intertemporal consistency of behavior. To gain some perspective, imagine you observe conditional preferences at every node and find that behavior is dynamically consistent. If behavior is also consequentialist, the individual necessarily observes and recognizes any event that has transpired. That is, she updates her ranking of actions in response to objective information. The consistency of her behavior then ‘reveals’ that the individual has foreseen all possible changes of the environment and incorporated them into her plans.

To extend the result in Theorem 6 to the dynamic model of limited foresight, a preliminary definition is necessary. Below, $\{\mathcal{G}^t_\omega\}$ denotes an $\mathcal{F}$-adapted process of filtrations.

**Definition 7** A family of posterior preferences is **dynamically consistent** relative to $\{\mathcal{G}^t_\omega\}$ if for each $t$ and $\omega$ and all $\mathcal{G}^t_\omega$-adapted acts $g, g'$:

$$g \succeq_{t+1,\omega} g' \text{ for all } \omega' \in \mathcal{F}_t(\omega) \implies g \succeq_{t,\omega} g'.$$

Define a filtration $\mathcal{G}$ to be **regular** if $\mathcal{G}_T = \mathcal{G}_{T-1}$. A process $\{\mathcal{G}^t_\omega\}$ is regular if $\mathcal{G}^t_\omega$ is regular for every $t$ and $\omega$.\(^7\) The process $\{\mathcal{G}^t_\omega\}$ refines the process $\{\mathcal{G}^{t,\omega}_t\}$ if $\mathcal{G}^{t,\omega}_t$ refines $\mathcal{G}^t_\omega$ for every $t$ and $\omega$.

The corollary below proves that the process of subjective filtrations $\{\mathcal{G}^{t,\omega}_t\}$ is the largest refining, regular process relative to which a limited foresight model $\{\succeq_{t,\omega}\}$ is dynamically consistent. The result provides an alternative characterization of the collection of foreseen events based on the intertemporal consistency of dynamic behavior.

**Corollary 7** If $\{\succeq_{t,\omega}\}$ is a dynamic model of limited foresight, then the process of foreseen filtrations $\{\mathcal{G}^{t,\omega}_t\}$ is the largest refining, regular process relative to which $\{\succeq_{t,\omega}\}$ is dynamically consistent.

\(^7\)By the richness assumption $\mathcal{F}_T = \mathcal{F}_{T-1}$, all sequentially connected filtrations are regular.
4 Related Literature

Kreps [13] is the first to model unforeseen contingencies. He takes as primitive of the model a state space $S$ depicting the individual’s incomplete perception of the environment. By modeling the preference for flexibility, Kreps derives an extended state space $\Omega := S \times \Theta$ and interprets the endogenous contingencies $\theta$ as completing the individual’s perception.

Both Kreps [13] and Dekel, Lipman, and Rustichini [3] observe that the model is observationally equivalent to a standard model with an extended state space $S \times \Theta$. In that interpretation, all events are foreseen but some contingencies, namely $\theta$, are unverifiable by an outside observer.

Gilboa and Schmeidler [10] reinterpret models of ambiguity about likelihoods as models of unforeseen contingencies. Like Kreps [13], they derive an extended state space $\Omega$ which completes the individual’s perception captured by the exogenous state space $S$. The nonadditivity of their model reflects the behavior of a self-aware person who tries to hedge against unanticipated contingencies.

Epstein, Marinacci, and Seo [4] argue against modeling the individual’s perception as an observable primitive. Modifying Kreps’ [12] framework of preference for flexibility, they endogenize both the coarse state space $S$ and its completion $\Omega$. However, their model retains the observational equivalence with ambiguity aversion.

5 Appendix

Throughout the appendix, $\mathcal{G}$ is a finitely generated algebra. The correspondix simplex $\Delta$ is endowed with the standard Euclidean topology and $\Delta^\circ$ denotes its interior.

A filtration $\{\mathcal{G}_t\}$ on a state space $\Omega$ is identified with the algebra $\mathcal{G}$ on $\Omega \times \mathcal{T}$ generated by the sets $A \times \{t\}$ for $A \in \mathcal{G}_t$ and $t \in \mathcal{T}$. Under this identification, an act $h$ is $\{\mathcal{G}_t\}$-adapted if and only if the mapping $(\omega, t) \mapsto h_t(\omega)$ is $\mathcal{G}$-measurable. The symbol $\mathcal{G}$ is used interchangeably to denote the algebra on $\Omega \times \mathcal{T}$ and the filtration $\{\mathcal{G}_t\}$ on $\Omega$.

For any $A \subset \Omega$, $x \in M$ and $f \in M^\Omega$, $f Ax := f' \in M^\Omega$ where $f'(\omega) = f(\omega)$ if $\omega \in A$, and $f'(\omega) = x$ if $\omega \in A^c$. For any $A \in \mathcal{F}_t$, $x \in M$ and $h \in \mathcal{H}$, $h Ax := (h_{-t}, h_t Ax)$. 

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5.1 Proof of Theorem 2

Adopt the arguments in Epstein and Schneider [5, Lemma A.1] to deduce that \( \succeq \) has a representation:

\[
U(h) = \min_{q \in Q} \sum_t \beta^t \langle q_t, h_t \rangle.
\]

(5.1)

Above, \( Q \) is a closed, convex subset of \( \times_{t \in T} \Delta(\Omega, F_t) \). Denote a generic element in \( Q \) by \( q := (q_t)_{t \in T} \). For every subset \( T' \subset T \), \( \text{proj}_{T'}(q) \) denotes the vector \( (q_t)_{t \in T'} \). Without loss of generality, set \( \beta = 1 \) and define \( \langle q, h \rangle := \sum_t \langle q_t, h_t \rangle \).

A subfiltration \( \mathcal{G} \) of \( F \) defines the following subspace of \( \times_{t \in T} ba(\Omega, F_t) \):

\[
\text{diag}(\mathcal{G}) := \{ q \in \times_{t \in T} ba(\Omega, F_t) : \text{marg}_{\mathcal{G}_t} q_{t+1} = \text{marg}_{\mathcal{G}_t} q_t \text{ for all } t < T \}.
\]

Note that \( \text{diag}(\mathcal{G}) \neq \text{diag}(\mathcal{G}') \) whenever \( \mathcal{G}_t \neq \mathcal{G}'_t \) for some \( t < T \). Equivalently, there exists a bijection between the diagonals of \( \times_{t \in T} ba(\Omega, F_t) \) and the set of subfiltrations \( \mathcal{G} \) such that \( \mathcal{G}_T = \mathcal{G}_{T-1} \). Call such subfiltrations regular.

For a regular subfiltration \( \mathcal{G} \), the following lemma establishes a basic duality between the diagonal \( \text{diag}(\mathcal{G}) \) and the set of effectively certain acts in \( \mathcal{H}_\mathcal{G} \). In view of (5.1) and after appropriate normalization, the latter can be identified with the subset:

\(
\mathcal{H}^c := \{ h \in \mathcal{H} : \sum_t h_t(\omega) = 0 \text{ for all } \omega \in \Omega \}.
\)

**Lemma 8** \( q \in \text{diag}(\mathcal{G}) \) if and only if \( \langle q, h \rangle = 0 \) for all \( h \in \mathcal{H}_\mathcal{G} \cap \mathcal{H}^c \).

**Proof:** To prove necessity observe that for any \( h \in \mathcal{H}_\mathcal{G} \) and \( q \in \text{diag}(\mathcal{G}) \),

\[
\langle q, h \rangle = \sum_t \langle q_t, h_t \rangle = \sum_t \langle q_{T'}, h_t \rangle = \langle q_T, \sum_t h_t \rangle.
\]

(5.2)

If \( h \in \mathcal{H}_\mathcal{G} \cap \mathcal{H}^c \), then \( \sum_t h_t(\omega) = 0 \) for all \( \omega \) and \( \langle q, h \rangle = \langle q_T, \sum_t h_t \rangle = 0 \) for all \( q \in \text{diag}(\mathcal{G}) \).

To establish sufficiency, first prove that for any \( h \in \mathcal{H}_\mathcal{G} \)

\[
h \in \mathcal{H}^c \text{ if and only if } \langle q, h \rangle = 0 \text{ for all } q \in \text{diag}(\mathcal{G}).
\]

(5.3)

Sufficiency of (5.3) follows (5.2). To see the reverse implication, fix some \( h \in \mathcal{H}_\mathcal{G} \setminus \mathcal{H}^c \) and without loss of generality suppose that \( \sum_t h_t(\omega) > 0 \) for
some \( \omega \). Let \( q_T \) be a measure in \( \Delta(\Omega, \mathcal{G}_T) \) such that \( q_T(\mathcal{G}_T(\omega)) = 1 \). Since \( h \in \mathcal{H}_G \), \( \langle q_T, \sum_i h_i \rangle \) is well-defined and strictly positive. Extend \( q_T \) to a vector \( q \in \text{diag}(\mathcal{G}) \) and note that \( \langle q, h \rangle = \langle q_T, \sum_i h_i \rangle > 0 \), proving the necessity of (5.3).

To complete the proof of the lemma, fix some \( q' \notin \text{diag}(\mathcal{G}) \). It suffices to find an act \( h' \in \mathcal{H}_G \cap \mathcal{H}^c \) such that \( \langle q', h' \rangle \neq 0 \). Since \( \text{diag}(\mathcal{G}) \) is a subspace of \( \times_{t \in T} \text{ba}(\Omega, \mathcal{G}_t) \) and any subspace is the intersection of the hyperplanes that contain it, there exists an act \( h' \in \mathcal{H}_G \) such that

\[
\langle q', h' \rangle \neq 0 \text{ and } \langle q, h' \rangle = 0 \text{ for all } q \in \text{diag}(\mathcal{G}).
\]

By (5.3), the act \( h' \) lies in \( \mathcal{H}_G \cap \mathcal{H}^c \), as desired.

**Lemma 9** For every closed set \( Q \subset \times_{t \in T} \Delta(\Omega, \mathcal{G}_t) \),

\[
Q \subset \text{diag}(\mathcal{G}) \text{ if and only if } \min_{q \in Q} \langle q, h \rangle = 0 \text{ for all } h \in \mathcal{H}_G \cap \mathcal{H}^c.
\]

**Proof:** Sufficiency follows directly from Lemma 8. To see necessity, suppose there exists some \( q' \in Q \setminus \text{diag}(\mathcal{G}) \). By Lemma 8, there exists \( h' \in \mathcal{H}_G \cap \mathcal{H}^c \) such that \( \langle q', h' \rangle \neq 0 \). Since

\[
h' \in \mathcal{H}_G \cap \mathcal{H}^c \text{ if and only if } -h' \in \mathcal{H}_G \cap \mathcal{H}^c,
\]

one can choose \( h' \) such that \( \min_{q \in Q} \langle q, h' \rangle \leq \langle q', h' \rangle < 0 \). This establishes a contradiction.

Define the subset of subjectively certain acts:

\[
\mathcal{H}^* := \{ h \in \mathcal{H}^c : h' \sim h'(\omega) \text{ for all } \omega \in \Omega \text{ and all } h' \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)} \},
\]

and let \( \mathcal{G}^* \) be the algebra on \( \Omega \times T \) induced by \( \mathcal{H}^* \).

**Lemma 10** The algebra \( \mathcal{G}^* \) on \( \Omega \times T \) is a regular filtration on \( \Omega \).

**Proof:** First prove that for every \( h \in \mathcal{H}^c \), the filtration \( \mathcal{F}(h) \) is the smallest algebra on \( \Omega \times T \) induced by \( \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)} \). For every \( t < T \), the act \( (0_{-t,-(t+1)}, 1_{\Omega}, -1_\Omega) \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{F}(h)} \) implying that the smallest algebra contains the set \( \Omega \times \{ t \} \) for every \( t \in T \). Also,

\[
h \in \mathcal{H}^c \text{ if and only if } h_T = -\sum_{\tau < T} h_\tau.
\]
Conclude that for every \( h \in \mathcal{H} \), \( \sigma(h_T) \leq \sigma(h_T) \). But then
\[
\mathcal{F}(h)_T = \sigma(h_T) = [\sigma(h_T)] \cup \sigma(h_T) = \sigma(h_T) = \mathcal{F}(h)_T.
\]
Conclude that \( \mathcal{F}(h) \) is regular and for all events \( A \in \mathcal{F}(h)_T \) and payoffs \( x \in M \) (in particular \( x \neq 0 \)):
\[
(0_{(T-1), -T}, xA^c(-x), xA(-x)) \in \mathcal{H} \cap \mathcal{H}_\mathcal{F}(h),
\]
and, since \( \mathcal{F}(h) \) is a filtration, for all \( t < T \) and \( A \in \mathcal{F}(h)_t \),
\[
(0_{-t, -(t+1)}, xA^c(-x), xA(-x)) \in \mathcal{H} \cap \mathcal{H}_\mathcal{F}(h).
\]
The above inclusions imply that \( \mathcal{F}(h) \) is the smallest algebra induced by \( \mathcal{H} \cap \mathcal{H}_\mathcal{F}(h) \).

Finally, fix \( h \in \mathcal{H}^* \), \( h' \in \mathcal{H} \cap \mathcal{H}_\mathcal{F}(h) \), and \( h'' \in \mathcal{H} \cap \mathcal{H}_\mathcal{F}(h') \). Since \( \mathcal{F}(h') \leq \mathcal{F}(h) \), \( h'' \in \mathcal{H} \cap \mathcal{H}_\mathcal{F}(h) \). By the choice of \( h \), the latter implies \( h'' \sim h''(\omega) \) for all \( \omega \). Conclude that \( h' \in \mathcal{H}^* \). But then
\[
\mathcal{H}^* = \bigcup_{h \in \mathcal{H}^*} [\mathcal{H} \cap \mathcal{H}_\mathcal{F}(h)] \Rightarrow \\
\mathcal{G}^* = \bigvee_{h \in \mathcal{H}^*} \mathcal{F}(h).
\]

Since the supremum of regular filtrations is a regular filtration, the lemma is proved.

**Lemma 11** \( Q \cap \text{diag}(\mathcal{G}^*) \neq \emptyset \).

**Proof:** Let \( Q' = \{ (\text{marg}_{G^*_t} q_t)_{t \in T} : (q_t)_{t \in T} \in Q \} \) and define the linear functional
\[
\phi : (q_t)_{t > 0} \mapsto (\text{marg}_{G^*_t} q_t)_{t > 0}.
\]
Consider the following subdomains of acts
\[
\begin{align*}
D_{T \setminus \{ T \}} & : = \{ (h_0, ..., h_{T-1}, x_0) \in \mathcal{H} \}, \\
D_{T \setminus \{ 0 \}} & : = \{ (x_0, h_0, ..., h_{T-1}) : (h_0, ..., h_{T-1}, x_0) \in \mathcal{H} \}.
\end{align*}
\]
Under the obvious identification, \( D_{T \setminus \{ T \}} = D_{T \setminus \{ 0 \}} \). The restrictions of \( \geq \) to \( D_{T \setminus \{ T \}} \) and \( D_{T \setminus \{ 0 \}} \), respectively, are represented by the following utility functions:
\[
\begin{align*}
U_{D_{T \setminus \{ T \}}} & = : \min_{q \in \text{proj}_{T \setminus \{ T \}} Q'} \sum_{t < T} (q_t, h_t) \\
U_{D_{T \setminus \{ 0 \}}} & = : \min_{q \in \phi \circ \text{proj}_{T \setminus \{ 0 \}} Q'} \sum_{t > 0} (q_t, h_t)
\end{align*}
\]
By Stationarity, $U_{D_{T\setminus\{T\}}}^{T}$ and $U_{D_{T\setminus\{0\}}}$ represent the same preference relation. [9, Theorem 1] implies that

$$\text{proj}_{T\setminus\{T\}} Q' = \phi \circ \text{proj}_{T\setminus\{0\}} Q' =: K$$

Define the correspondence

$$\psi := \phi \circ \text{proj}_{T\setminus\{0\}} \circ \left( Q' \cap \text{proj}^{-1}_{T\setminus\{T\}} \right) : K \rightarrow K$$

Since $Q'$ is closed, $\psi$ is the composition of a continuous function and an upper hemi-continuous correspondence. Thus $\psi$ is upper hemi-continuous. Since $Q'$ is convex, $\psi$ is also convex-valued. By the Kakutani fixed point theorem [1, Corollary 16.51], $\psi$ has a fixed point $q \in \psi(q)$. Equivalently, there exists a point $(q_0, q_1, ..., q_T) \in Q'$ such that

$$\phi(q_1, ..., q_T) = (q_0, q_1, ..., q_{T-1})$$
$$\Leftrightarrow$$
$$\text{marg}_{G_{t-1}} q_t = q_{t-1}, \forall t > 1.$$  

**Lemma 12** $\mathcal{H}^c \cap \mathcal{H}_{G^*} = \mathcal{H}^*$ and $\mathcal{G}^*$ is the largest regular filtration $\mathcal{G}$ such that $Q \subset \text{diag}(\mathcal{G})$.

**Proof:** By construction, $\mathcal{H}^* \subset \mathcal{H}^c \cap \mathcal{H}_{G^*}$. To see the reverse inclusion, fix $h \in \mathcal{H}^c \cap \mathcal{H}_{G^*}$ and let $x \sim (-h)$. Strong Certainty Independence implies that

$$\frac{1}{2} x + \frac{1}{2} h \sim \frac{1}{2} (-h) + \frac{1}{2} h.$$ The two indifferences imply

$$x = -\frac{1}{T+1} \max_{q \in Q} \sum_t \langle q_t, h_t \rangle,$$
$$x = -\frac{1}{T+1} \min_{q \in Q} \sum_t \langle q_t, h_t \rangle.$$

Conclude that for every $h \in \mathcal{H}^c \cap \mathcal{H}_{G^*}$, $\langle q, h \rangle = \langle q', h \rangle$ for every $q, q' \in Q$. By Lemma 11, there exists a $q \in Q \cap \text{diag}(\mathcal{G}^*)$. It follows that $\langle q, h \rangle = 0$ for every $q \in Q$ and so $\mathcal{H}^c \cap \mathcal{H}_{G^*} \subset \mathcal{H}^*$. Also by Lemma 9, $Q \subset \text{diag}(\mathcal{G}^*)$.

If $\mathcal{G}$ is any filtration such that $G_T = G_{T-1}$ and $Q \subset \text{diag}(\mathcal{G})$, Lemma 9 implies that

$$\mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}} \subset \mathcal{H}^* = \mathcal{H}^c \cap \mathcal{H}_{G^*}.$$

Conclude that $\mathcal{G} \subset \mathcal{G}^*$.
5.1.1 Properties of the Filtration $\mathcal{G}^*$

Lemma 13 $\mathcal{G}^*$ is connected, that is, $\mathcal{G}^*_t = \mathcal{G}^* \cap \mathcal{F}_t$ for all $t \in T$.

Proof: By Lemma 10, $\mathcal{G}^*$ is a filtration. Thus, $\mathcal{G}^*_t \subseteq \mathcal{G}^* \cap \mathcal{F}_t$ for all $t$. Conversely, fix an event $A \in \mathcal{G}^*_t \cap \mathcal{F}_t$ for some $t \in T$. Since $\mathcal{G}^*$ is regular by Lemma 10, $\mathcal{G}^*_t \supseteq A$ for $t = T - 1$. For $t < T - 1$, it suffices to show that $(0_{-t,-(t+1)}, xAy, (-x)A(-y)) \sim 0$ for all $x, y \in M$. By the regularity of $\mathcal{G}^*$, $(0_{-(T-1),-T}, xAy, (-x)A(-y)) \in \mathcal{H}^c \cap \mathcal{H}_{\mathcal{G}^*}$, which implies that

$$(0_{-(T-1),-T}, xAy, (-x)A(-y)) \sim 0.$$ 

Applying Stationarity repeatedly, conclude that

$$(0_{-t,-(t+1)}, xAy, (-x)A(-y)) \sim 0. \blacksquare$$

Lemma 14 $\mathcal{G}^*$ is sequentially connected.

Proof: Fix an event $A \in \Pi_{\mathcal{G}_t} \setminus \Pi_{\mathcal{F}_t}$ for some $t < T$. By way of contradiction, suppose there exists a set $\emptyset \neq B \in \mathcal{G}_{t+1}$ such that $B \subsetneq A$. First, suppose $B \subset C \subsetneq A$ for some $C \in \Pi_{\mathcal{G}_t}$. Since $\mathcal{G}$ is connected, conclude that $B \subsetneq C$. Otherwise, $B = C \in \mathcal{G}_{t+1} \cap \Pi_{\mathcal{F}_t}$ implies that $C \in \Pi_{\mathcal{G}_t}$, contradicting the choice of $A$. Now take the acts $g = (0_{-t,-(t+1)}, 1_A, 1_B)$ and $g' = (0_{-t,-(t+1)}, 1_A, 1_C)$.

By construction, $g \in \mathcal{H}_{\mathcal{G}}$ and $g'$ simplifies $g$ at $C \in \Pi_{\mathcal{F}_t}$. By Sequentiality, $g' \in \mathcal{H}_{\mathcal{G}}$ and so $C \in \mathcal{G}_{t+1} \cap \Pi_{\mathcal{F}_t}$. Since $\mathcal{G}$ is connected, $C \in \Pi_{\mathcal{G}_t}$ contradicting $C \subsetneq A \in \Pi_{\mathcal{G}_t}$.

Conversely, suppose $B \cap C \neq \emptyset$ and $B \cap C^c \neq \emptyset$ for some $C \in \Pi_{\mathcal{F}_t}$. The act $g = (0_{-t,-(t+1)}, 1_A, 1_B)$ is $\mathcal{G}^*$-measurable. Moreover, the continuation of $g' := (0_{-t,-(t+1)}, 1_A, 1_{B \cup C})$ at the node $C \in \Pi_{\mathcal{F}_t}$ simplifies the continuation of $g$. By Sequentiality, $B \cup C \in \mathcal{G}_{t+1}$. But $B \cup C \in \mathcal{G}_{t+1}$ and $B \in \mathcal{G}_{t+1}$ imply $C \setminus B = B \cup C \setminus B \in \mathcal{G}_{t+1}$ and, by construction, $\emptyset \neq C \setminus B \subsetneq C \subsetneq A$ and $C \in \Pi_{\mathcal{F}_t}$. But then $g' = (0_{-t,-(t+1)}, 1_A, 1_C)$ simplifies $g = (0_{-t,-(t+1)}, 1_A, 1_{C \setminus B}) \in \mathcal{H}_{\mathcal{G}}$ at $C \in \Pi_{\mathcal{F}_t}$. As before, conclude that $C \in \Pi_{\mathcal{G}_t}$ contradicting $C \subsetneq A \in \Pi_{\mathcal{G}_t}$.

5.1.2 Construction of the Approximation Mapping $\Phi$

Lemma 15 For all acts $h, h'$, payoffs $x$ and $y$ and events $A \in \cup_t \Pi_{\mathcal{G}_t}$,

$$hAx \succeq h'Ax \text{ if and only if } hAy \succeq h'Ay$$
**Proof:** Suppose $hAx \succeq h'Ax$ and note that $(hAy)Ax = hAx$ and $(h'Ay)Ax = h'Ax$. Conclude that $(hAy)Ax \succeq (h'Ay)Ax$. For any $A' \in \cup_i \Pi_{q_i^*}$ and $A' \neq A$,

$$(hAy)A'x = (h'Ay)A'x = yA'x.$$ Conclude that

$$(hAy)A'x \sim (h'Ay)A'x.$$ Thus, $(hAy)A'x \succeq (h'Ay)A'x$ for all $A' \in \cup_i \Pi_{q_i^*}$. By Subjective Monotonicity, $hAy \succeq h'Ay$ as desired.

Fix some $x_0$ and for every $A \in \cup_i \Pi_{q_i^*}$, define the preference $\succeq_A$ as

$$h \succeq_A h' \text{ if and only if } hAx_0 \succeq h'Ax_0.$$ By the above lemma, the ‘conditional’ preference $\succeq_A$ is independent of the choice of $x_0$. By construction, $\succeq_A$ inherits convexity, monotonicity and mixture-continuity. By Nonnullity, the preference is also nontrivial. By Strong Certainty Independence and by Lemma 15 in turn,

$$hAx_0 \succeq h'Ax_0 \Rightarrow [\alpha h + (1 - \alpha)x]A[\alpha x_0 + (1 - \alpha)x] \succeq [\alpha h' + (1 - \alpha)x]A[\alpha x_0 + (1 - \alpha)x] \Rightarrow [\alpha h + (1 - \alpha)x]Ax_0 \succeq [\alpha h' + (1 - \alpha)x]Ax_0$$ Conclude that for all $A \in \cup_i \Pi_{q_i^*}$, $\succeq_A$ is a multiple prior preference. Recall that, for any tuple $(\omega, t) \in \Omega \times T$, $G_i^*(\omega)$ denotes the event in $\Pi_{q_i^*}$ containing $\omega$. By [9, Theorem 1], there exists a set $C_{\omega, t} \subset \Delta(G_i^*(\omega), F_t)$ such that

$$h \succeq_{q_i^*(\omega)} h' \text{ if and only if } \min_{q \in C_{\omega, t}} \langle q, h_t \rangle \geq \min_{q \in C_{\omega, t}} \langle q, h_t' \rangle.$$ (5.4) Define the mapping $\Phi : \mathcal{H} \rightarrow \mathcal{H}$

$$(\Phi h)_t(\omega) = \min_{q \in C_{\omega, t}} \langle q, h_t \rangle.$$ By construction,

$$\Phi(hAx_0) = \Phi(h)Ax_0 \sim hAx_0 \text{ for all } A \in \cup_i \Pi_{q_i^*}.$$ By Subjective Monotonicity, $\Phi(h) \sim h$ for all $h \in \mathcal{H}$. It is evident that $\Phi$ is an approximation mapping. To conclude the proof of the theorem, for all
\[ h \in \mathcal{H}, \text{ define} \]
\[
V(h) = U \circ \Phi(h) \\
= \min_{q \in Q} \sum_t \langle q_t, \Phi(h)_t \rangle \\
= \min_{q \in Q} \sum_t \langle q_T, \Phi(h)_t \rangle \\
= \min_{q \in Q} \langle q_T, \sum_t \Phi(h)_t \rangle.
\]

The third equality follows from Lemma 12. Finally set
\[
\mathcal{C} := \text{marg}_{G_T} \circ \text{proj}_{\{T\}} Q.
\]

The claim that \( \mathcal{C} \) is a subset of \( \Delta^\circ(\Omega, \mathcal{G}_T^*) \) follows from the following property of multiple-prior preferences.

**Lemma 16** Let \( \succeq \) be a multiple-prior preference on \( \mathcal{F}_T \)-measurable functions in \( M^\Omega \), and let \( \mathcal{C} \) be the respective set of priors. Let \( \Pi \) be any partition such that \( \Pi \leq \Pi_{\mathcal{F}_T} \). If every event \( A \in \Pi \) is nonnull and if for all payoffs \( x \in M \):

\[
hAx \succeq h'Ax \text{ for all } A \in \Pi \text{ implies } h \succeq h'
\]

then \( p(A) > 0 \) for all \( A \in \Pi \) and \( p \in \mathcal{C} \).

**Proof:** Suppose by way of contradiction that \( p(A) = 0 \), for some \( A \in \Pi \). Since \( A \) is nonnull, \( \max_{q \in \mathcal{C}} q(A) > p(A) = 0 \). Fix some payoffs \( y, y' \) such that \( 1 > y > y' > 0 \) and note that:

\[
U(yA0) = \min_{q \in \mathcal{C}} [q(A)y] = y \min_{q \in \mathcal{C}} q(A) = 0 = U(y'A0), \text{ and}
\]
\[
U(yA1) = \min_{q \in \mathcal{C}} [(y - 1)q(A) + 1] = 1 - (1 - y) \max_{q \in \mathcal{C}} q(A) >
\]
\[
> 1 - (1 - y') \max_{q \in \mathcal{C}} q(A) = U(y'A1).
\]

Conclude that \( yA0 \sim y'A0 \) and \( yA1 \succ y'A1 \) in contradiction of Lemma 15. \( \blacksquare \)
5.1.3 Uniqueness

Uniqueness of the set $C$ follows from familiar arguments. To prove the uniqueness of the $G^*$-approximation mapping, take two such mappings $\Phi$, $\tilde{\Phi}$. By separability, for all $A \in \Pi_{G^*}$, $h'$ and $x$:

$$\tilde{\Phi}(h'Ax) = \tilde{\Phi}((h'Ax)Ax)A\tilde{\Phi}((h'Ax)A'x)$$
$$= \tilde{\Phi}(h'Ax)A\tilde{\Phi}(x)$$
$$= \tilde{\Phi}(h'Ax)Ax$$

The last equality follows from the fact that $\tilde{\Phi}$ must be identity on $G^*$-measurable acts. Since $A \in \bigcup \Pi_{G^*_t}$ and $\Phi(h'Ax)$, $\tilde{\Phi}(h'Ax) \in H_{G^*}$, there exist payoffs $y_\Phi$, $y_{\tilde{\Phi}}$ such that

$$\tilde{\Phi}(h'Ax) = y_{\tilde{\Phi}}Ax$$
$$\Phi(h'Ax) = y_\Phi Ax$$

Since $\tilde{\Phi}(h'Ax) \sim \Phi(h'Ax)$ and $\succeq$ is strictly increasing on $H_{G^*}$, it must be the case that $y_\Phi = y_{\tilde{\Phi}}$. The proof is completed by induction on the number of events $A \in \Pi_{G^*}$ such that an act $h' \in H$ is nonconstant.

5.2 An Alternative Formulation

This section describes an alternative formulation of the static model.

**Definition 8** A preference relation $\succeq$ on $H$ has a **regular representation** $(G, \Phi, C)$ if it admits a utility function $V$ of the form (2.3) where $G$ is regular, the mapping $\Phi$ is identity on $H_G$ and $C$ is a closed, convex subset of $\Delta^\circ(\Omega, G_T)$.

**Definition 9** A preference relation $\succeq$ on $H$ has a **largest representation** $(G, \Phi, C)$ if it admits a utility function $V$ of the form (2.3) where $G$ is sequentially connected, $\Phi$ is a $G$-approximation mapping, $C$ is a closed, convex subset of $\Delta^\circ(\Omega, G_T)$ and $G$ is the largest filtration for which a regular representation exists.

**Lemma 17** $(G, \Phi, C)$ is a limited foresight representation if and only if it is a largest representation.
Proof: If \((G, \Phi, C)\) is a regular representation for \(\succeq\), then \(H^c \cap H_G \subset H^*\) and so \(G \subset G^*\). Thus, the limited foresight model is sufficient for a largest representation. Conversely, if a largest representation \((G^*, \Phi, C)\) exists, then

\[
G^* = \bigvee_{Q \in \text{diag}(G')} G' \Rightarrow H^c \cap H_{G^*} = \bigcup_{(g' : Q \in \text{diag}(g'))} H^c \cap H_{G'}
\]

At the same time,

\[
H^* = \bigcup_{(g' : Q \in \text{diag}(g'))} H^c \cap H_{G'}.
\]

To see this note that, \(\bigcup_{(g' : Q \in \text{diag}(g'))} H^c \cap H_{G'} \subset H^*\). If \(h \in H^*\), then for all \(h' \in H^c \cap H_{F(h)}\), \(h' \sim 0\). From Lemma 9, conclude that \(Q \subset \text{diag}(F(h))\) and so \(h \in \bigcup_{g' : Q \subset \text{diag}(g')} H^c \cap H_{G'}\). Thus \(H^* = H^c \cap H_{G^*}\) and so \(G^* = F(H^*)\) as desired.

### 5.3 Proof of Lemma 3

Suppose \(\succeq\) has a representation \((G, \{C_A\}, C)\) such that \(C_A\) has nonempty interior in \(\Delta(A, \mathcal{F}_t)\) for all \(t \in T\) and \(A \in \Pi_G\). Let \(Q\) be the set in \(\times_t \Delta(\Omega, \mathcal{F}_t)\) that represents \(\succeq\) as in (5.1). It suffices to show that for all filtrations \(G'\) such that \(G'_{T-1} = G'_t\):

\[
Q \subset \text{diag}(G') \text{ implies } G' \leq G.
\]

Equivalently,

\[
\forall t < T, \forall B \notin G_t, \exists q \in Q \text{ such that } q_t(B) \neq q_{t+1}(B).
\]

For all \(t \in T\) and \(A \in \Pi_G\), define \(\succeq_A\) as in (5.4). The family of preferences \(\{\succeq_A\}\) and \(\succeq\) satisfy the conditions of [5, Theorem 3.2]. Conclude that:

\[
Q = \bigcup_{\mu \in C} \{q_t\} : q_t = \int p_A d\mu \text{ for some selection } \{p_A\}_{A \in \Pi_G} \text{ s.t. } p_A \in C_A.
\]

Fix \(t < T\) and \(B \notin G_t\) and any \(\mu \in C\). By the above decomposition of \(Q\), it suffices to find two selections \(\{p_A\}_{A \in \Pi_G}\) and \(\{p'_A\}_{A \in \Pi_G}\) such that:

\[
\int p_A(B) d\mu \neq \int p'_A(B) d\mu.
\]

Since \(B \notin G_t\), there exists \(A^* \in \Pi_G\) such that \(A^* \neq B \cap A^* \neq \emptyset\). Since \(C_{A^*}\) has nonempty interior, there exist \(p_{A^*}\) and \(p'_{A^*}\) such that \(p_{A^*}(B) \neq p'_{A^*}(B)\).

Choose any \(p_A = p'_A\) for all \(A \neq A^*\) and \(A \in \Pi_G\) to complete the proof of the lemma.
5.4 Proof of Theorem 4

Necessity is standard. To prove sufficiency, first show that \( \mathcal{G}^{t,\omega} \) is refining. Fix \( t, \omega \) such that \( \mathcal{F}_{t+1}(\omega) \notin \mathcal{G}^{t,\omega} \). Since \( \mathcal{G}^{t,\omega} \) is sequentially connected:

\[
\{B \cap \mathcal{F}_{t+1}(\omega) : B \in \mathcal{G}^{t,\omega}\} = \{\emptyset, \mathcal{F}_{t+1}(\omega)\}.
\]

Conclude that \( \mathcal{G}^{t+1,\omega} \) refines the trivial filtration \( \mathcal{G}^{t,\omega} \cap \mathcal{F}_{t+1}(\omega) \). Next, suppose \( \mathcal{F}_{t+1}(\omega) \in \mathcal{G}^{t,\omega} \) and take an act \( h \in \mathcal{H}_{G^{t,\omega} \cap \mathcal{H}^c} \) such that \( h_{t'}(\omega') = 0 \) whenever \( t' < t \) or \( \omega' \notin \mathcal{F}_{t+1}(\omega) \). By Consequentialism, it suffices to show that \( h \) is indifferent to the constant act \( 0 \). By construction, \( h \sim_{t+1,\omega'} 0 \) for all \( \omega' \notin \mathcal{F}_{t+1}(\omega) \). Since \( h, 0 \in \mathcal{H}_{G^{t,\omega}} \), Weak Dynamic Consistency and Lemma 15 imply that:

\[
h \sim_{t+1,\omega'} 0 \quad \text{if and only if} \quad h \sim_{t,\omega} 0.
\]

The latter indifference is true by the choice of \( h \in \mathcal{H}_{G^{t,\omega} \cap \mathcal{H}^c} \).

To prove that \( \{\mathcal{C}^{t,\omega}\} \) admits a consistent extension, construct the set \( \mathcal{C} \) recursively. For all \( \omega \) and \( t \geq T-1 \), set \( \mathcal{C}^{t,\omega} := \mathcal{C}^{t,\omega} \). Fix \( \omega \) and \( t < T-1 \) and suppose \( \mathcal{C}^{t+1,\omega'} \) has been defined for all \( \omega' \). For any \( \omega' \) such that \( \mathcal{F}_{t+1}(\omega') \notin \mathcal{G}^{t,\omega} \), fix some measure \( \lambda_{\omega'} \in \Delta^o(\mathcal{G}^{t,\omega}(\omega'), \mathcal{F}_{t+1}) \) such that \( \lambda_{\omega''} = \lambda_{\omega'} \) for all \( \omega'' \in \mathcal{G}^{t,\omega}(\omega') \). For each \( \mu \in \mathcal{C}^{t,\omega} \) define the measure \( \hat{\mu} := \int_{\Omega} \hat{\lambda}_{\omega'} dm \) in \( \Delta^o(\mathcal{F}(\omega), \mathcal{F}_{t+1}) \) where

\[
m := \text{marg}_{\mathcal{G}^{t+1,\omega}} \mu \quad \text{and} \quad \hat{\lambda}_{\omega'} := \begin{cases} \lambda_{\omega'} & \text{if} \ \mathcal{F}_{t+1}(\omega') \notin \mathcal{G}^{t,\omega} \\ \delta_{\omega'} & \text{if} \ \mathcal{F}_{t+1}(\omega') \in \mathcal{G}^{t,\omega} \end{cases}.
\]

In effect, the constructed measures \( \mu \) extend the individual’s one-step ahead beliefs at \( t, \omega \) from the foreseen events in \( \mathcal{G}^{t,\omega} \) to all \( \mathcal{F}_{t+1} \)-measurable subsets of \( \mathcal{F}(\omega) \). The construction ensures that the set of extensions \( M^{t,\omega} := \{\hat{\mu} : \mu \in \mathcal{C}^{t,\omega}\} \subset \Delta^o(\mathcal{F}(\omega), \mathcal{F}_{t+1}) \) is closed and convex.

Next, let \( p \) denote a generic, \( \mathcal{F}_{t+1} \)-measurable selection from \( \omega' \longmapsto \mathcal{C}^{t+1,\omega'} \) and define

\[
\mathcal{C}^{t,\omega} = \{\int_{\Omega} p_{\omega'} d\hat{\mu}(\omega') : \hat{\mu} \in M^{t,\omega} \quad \text{and} \quad p_{\omega'} \in \mathcal{C}^{t+1,\omega'} \quad \text{for all} \ \omega'\}.
\]

From [5, Theorem 3.2], conclude that \( \mathcal{C}^{t,\omega} \) is a closed and convex subset of \( \Delta^o(\mathcal{F}(\omega), \mathcal{F}_{t}) \) and \( \mathcal{C} := \mathcal{C}^{0} \) is \( \{\mathcal{F}_{t}\} \)-rectangular. In particular,

\[
\{\mu(\cdot | \mathcal{F}_{t}(\omega)) : \mu \in \mathcal{C}\} = \mathcal{C}^{t,\omega} \quad \text{for all} \ t \ \text{and} \ \omega.
\]

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To complete the proof, it remains to show that
\[ \text{marg}_{G^t, \omega} C^{t, \omega} := \{ \text{marg}_{G^t, \omega} \mu : \mu \in C^{t, \omega} \} = C^{t, \omega} \text{ for all } t \text{ and } \omega. \] (5.7)

The next lemmas show that both \( \text{marg}_{G^t, \omega} C^{t, \omega} \) and \( C^{t, \omega} \) admit decompositions similar to (5.6).

**Lemma 18** For all \( t \) and \( \omega \), \( \text{marg}_{G^t, \omega} C^{t, \omega} \) admits the decomposition
\[ \text{marg}_{G^t, \omega} C^{t, \omega} = \{ \int_\Omega p_{\omega'} d\mu : \mu \in \text{marg}_{G^t, \omega} M^{t, \omega} \text{ and } p_{\omega'} \in \text{marg}_{G^t, \omega} C^{t+1, \omega'} \}. \]

**Proof:** By (5.6), all measures in \( C^{t, \omega} \) are of the form \( M^{t, \omega} \) and \( p \) is an \( F_{t+1} \)-measurable selection from \( \omega' \mapsto \hat{C}^{t+1, \omega'} \). Since \( \hat{C}^{t, \omega} \) is sequentially connected, \( \text{marg}_{G^t, \omega} C^{t, \omega} \) admits a decomposition similar to (5.6). Let\( m \in \text{marg}_{G^t, \omega} C^{t, \omega} \) and \( p_{\omega'} \in \text{marg}_{G^t, \omega} C^{t+1, \omega'} \).

**Lemma 19** For all \( t \) and \( \omega \), \( C^{t, \omega} \) admits the decomposition
\[ C^{t, \omega} = \{ p_{\omega'} dm : m \in \text{marg}_{G^t, \omega} C^{t, \omega} \text{ and } p_{\omega'} \in \text{marg}_{G^t, \omega} C^{t+1, \omega'} \text{ for all } \omega' \}. \]

**Proof:** For each \( \omega' \in F_t(\omega) \), let \( \geq_{t+1, \omega'}^{a} \) and \( \geq_{t, \omega}^{a} \) denote the respective restrictions of \( \geq_{t+1, \omega'} \) and \( \geq_{t, \omega} \) to \( H^{G_{t+1}}_{\omega} \). Since \( H^{G_{t+1}}_{\omega} \) refines \( H^{G_{t}}_{\omega} \), for each \( \omega' \in F_t(\omega) \), the corresponding preference \( \geq_{t+1, \omega'}^{a} \) has a representation
\[ U^{t+1, \omega'}(h) = \min_{\mu \in \text{marg}_{G^t, \omega} C^{t+1, \omega'}} \int \sum_{\tau \geq \nu} \beta^{\tau-t-1} h_{\tau} d\mu. \]

Since \( G^{t, \omega} \) is sequentially connected, the mapping \( \omega' \mapsto \geq_{t+1, \omega'}^{a} \) is \( G^{t+1, \omega} \)-measurable. Thus the collection of preferences \( \geq_{t, \omega}, \geq_{t+1, \omega'} \) for \( \omega' \in F_t(\omega) \) satisfies Consequentialism with respect to the filtration \( G^{t, \omega} \). By State Independence and Lemma 15, the collection of preferences is also dynamically consistent. The claim of the lemma follows from [5, Theorem 3.2].

Complete the proof of (5.7) by induction. The claim holds trivially for \( \omega \) and \( t \geq T - 1 \). Fix some \( \omega \) and \( t < T - 1 \) and suppose the claim has
been established for $t + 1$. Applying Lemma 19, the induction hypothesis and Lemma 18 in turn, conclude that

$$\{\mu(\cdot | F_{t+1}(\omega')) : \mu \in \mathcal{C}_{t+1,\omega}'\} = \text{marg}_{G_{t,\omega}} \mathcal{C}_{t+1,\omega}'$$  \hspace{1cm} (5.8)$$

$$= \text{marg}_{G_{t,\omega}} \tilde{\mathcal{C}}_{t+1,\omega}'$$

$$= \text{marg}_{G_{t,\omega}} \{\mu(\cdot | F_{t+1}(\omega')) : \mu \in \tilde{\mathcal{C}}_{t,\omega}\}$$

Also, by construction,

$$\text{marg}_{G_{t+1}} \mathcal{C}_{t,\omega} = \text{marg}_{G_{t+1}} M_{t,\omega}.$$  \hspace{1cm} (5.9)$$

Properties (5.8) and (5.9) show that $\text{marg}_{G_{t,\omega}} \mathcal{C}_{t,\omega}$ and $\mathcal{C}_{t,\omega}$ induce the same sets of conditionals and one-step-ahead marginals. By Lemmas 18 and 19, both $\text{marg}_{G_{t,\omega}} \mathcal{C}_{t,\omega}$ and $\mathcal{C}_{t,\omega}$ are uniquely determined by the respective sets of conditionals and marginals. Conclude that $\text{marg}_{G_{t,\omega}} \mathcal{C}_{t,\omega} = \mathcal{C}_{t,\omega}$.

### 5.4.1 Uniqueness

Let $\mathcal{C}$ be an $\{F_t\}$-rectangular subset of $\Delta^\omega(\Omega, \mathcal{F}_T)$ and for each $t, \omega$, let $G_{t,\omega}$ be a sequentially connected algebra such that $F_{t+1}(\omega') \in G_{t,\omega}$ for all $\omega' \in F_t(\omega)$.

**Lemma 20** A measure $\mu$ in $\Delta^\omega(\Omega, \mathcal{F}_T)$ belongs to $\mathcal{C}$ if and only if

$$\text{marg}_{G_{t,\omega}} \mu(\cdot | F_t(\omega)) \in \{\text{marg}_{G_{t,\omega}} \mu'(\cdot | F_t(\omega)) : \mu' \in \mathcal{C}\}$$  \hspace{1cm} \text{for all } t \text{ and } \omega.$$

**Proof:** Sufficiency is immediate. To prove necessity, note the recursive construction of the $\{F_t\}$-rectangular set in the proof of Theorem 4. An $\{F_t\}$-rectangular subset contains the measure $\tilde{\mu}$ if and only if

$$\tilde{\mu}(\cdot | F_T(\omega)) \in \{\mu'|\cdot | F_T(\omega)) : \mu' \in \mathcal{C}\} \text{ for } \omega \in \Omega, \text{ and}$$

$$\tilde{\mu}(\cdot | F_t(\omega)) \in \{\text{marg}_{F_{t+1}} \mu'(\cdot | F_t(\omega)) : \mu' \in \mathcal{C}\} \text{ for } \omega \in \Omega \text{ and } t < T.$$  

By construction, the restriction of $F_T$ to $F_T(\omega)$ equals $\{F_T(\omega), \emptyset\}$ which equals $G_{T,\omega}$ for each $\omega \in \Omega$. By hypothesis, the restriction of $F_{t+1}$ to $F_t(\omega)$ is refined by $G_{t,\omega}$ for each $t$ and $\omega$. Conclude that $\mu \in \mathcal{C}$.}$

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To prove Theorem 5, let \( \{C^t, \omega\} \) be an \( \mathcal{F} \)-adapted process where \( C^t, \omega \subset \Delta^\circ(\mathcal{F}_t(\omega), \mathcal{G}^t, \omega) \). Say that the set \( C' \) in \( \Delta(\Omega, \mathcal{F}_T) \) extends \( \{C^t, \omega\} \) if
\[
\{\text{marg}_{\mathcal{G}^t, \omega} \mu(\cdot | \mathcal{F}_t(\omega)) : \mu \in C', \mu(\mathcal{F}_t(\omega)) > 0\} = C^t, \omega.
\]
It is not difficult to see that any extension \( C' \) must be a subset of \( \Delta^\circ(\Omega, \mathcal{F}_T) \).

By way of contradiction suppose that there exists a measure \( \mu^0 \in C' \) such that \( \mu^0(\mathcal{F}_t(\omega)) = 0 \) for some \( t \) and \( \omega \). Since for all \( \omega' \), \( \mu^0(\mathcal{F}_0(\omega')) = \mu^0(\Omega) = 1 \), conclude that \( t > 0 \). Let \( t^* \) be the largest \( t' \) such that \( \mu^0(\mathcal{F}_{t'}(\omega)) > 0 \). The time \( t^* \) exists since \( \mu^0(\mathcal{F}_0(\omega)) > 0 \). By the definition of \( t^* \), \( \mu^0(\cdot | \mathcal{F}_{t^*}(\omega)) \) is well-defined and \( \mu^0(\mathcal{F}_{t^*+1}(\omega) | \mathcal{F}_{t^*}(\omega)) = 0 \). The latter gives a contradiction, since \( \mathcal{F}_{t^*+1}(\omega) \in \mathcal{G}^{t^*, \omega} \) and
\[
\text{marg}_{\mathcal{G}^{t^*, \omega}} \mu^0(\cdot | \mathcal{F}_{t^*}(\omega)) \in \mathcal{C}_{t^*, \omega} \subset \Delta^\circ(\mathcal{F}_{t^*}(\omega), \mathcal{G}^{t^*, \omega}).
\]
From the recursive construction in the proof of Theorem 4, conclude that there exists an \( \{\mathcal{F}_t\} \)-rectangular extension \( C \in \Delta^\circ(\Omega, \mathcal{F}_T) \). Lemma 20 proves that \( C \) is the unique \( \{\mathcal{F}_t\} \)-rectangular extension, whenever \( \mathcal{F}_{t+1}(\omega') \in \mathcal{G}^{t, \omega} \) for all \( \omega' \in \mathcal{F}_t(\omega) \) and all \( t \) and \( \omega \). Moreover, any other, possibly non-rectangular, extension \( C' \) is a subset of \( C \).

### 5.5 Proof of Theorem 6

To prove Theorem 6, take an act \( h \) such that \( h(\omega) \sim_0 h(\omega') \) for all \( \omega, \omega' \in \Omega \).

By Consequentialism,
\[
h \sim_{T, \omega} h(\omega) \quad \text{for all } \omega \in \Omega.
\]
Fix some \( \omega \) and note that:
\[
h_{T}(\omega') = h_{T}(\omega'') \quad \text{for all } \omega', \omega'' \in \mathcal{F}_{T-1}(\omega) \text{ and } T \leq T - 1.
\]
By State Independence,
\[
h(\omega') \sim_{T, \omega} h(\omega'') \quad \text{for all } \omega', \omega'' \in \mathcal{F}_{T-1}(\omega).
\]
But then (5.10) and (5.11) imply that for all \( \omega' \in \mathcal{F}_{T-1}(\omega) \):
\[
h \sim_{T, \omega} h(\omega') \sim_{T, \omega'} h(\omega).
\]
By Dynamic Consistency, \( h \sim_{T-1, \omega} h(\omega) \). Proceeding inductively, conclude that \( h \sim_0 h(\omega) \).
5.6 Sequentially Connected Filtrations

Say that a filtration $\{\mathcal{G}_t\}$ is connected if

$$\mathcal{G}_t = \mathcal{F}_t \cap \mathcal{G}_T$$

for all $t \in \mathcal{T}$.

**Proposition 21** A sequentially connected filtration $\{\mathcal{G}_t\}$ is connected.

Let $\{\mathcal{G}_t\}$ be sequentially connected. It is evident that $\mathcal{G}_t \subset \mathcal{G}_T \cap \mathcal{F}_t$ for all $t \in \mathcal{T}$. To prove the opposite inclusion, take an event $A \in \mathcal{G}_T \cap \mathcal{F}_t$. If $A \notin \mathcal{G}_t$, then there exists a set $B \in \Pi_{\mathcal{G}_t}$ such that $\emptyset \neq B \cap A \neq B$. Then $A \in \mathcal{F}_t$ implies $B \notin \Pi_{\mathcal{F}_t}$. Conclude that $B \in \Pi_{\mathcal{G}_t} \setminus \Pi_{\mathcal{F}_t}$ and since $\{\mathcal{G}_t\}$ is connected, $B \in \Pi_{\mathcal{G}_T}$. But then $\emptyset \neq B \cap A \neq B$ contradicts the fact that $A \in \mathcal{G}_T$.

The following proposition shows that sequentially connected filtrations inherit the lattice properties of stopping-times.

**Proposition 22** The class of sequentially connected filtrations is a lattice. It is lattice-isomorphic to the class of sequentially connected algebras.

First, establish the following distributive law.

**Lemma 23** If $\mathcal{G}$ and $\mathcal{G}'$ are sequentially connected algebras, then

$$\Pi_{\mathcal{G} \cap \mathcal{F}_t} \lor \Pi_{\mathcal{G}' \cap \mathcal{F}_t} = \Pi_{\mathcal{G} \lor \mathcal{G}'} \land \Pi_{\mathcal{F}_t}$$

for all $t \in \mathcal{T}$.

**Proof:** For any partitions $\Pi$ and $\Pi'$, $\Pi \subset \Pi'$ if and only if $\Pi = \Pi'$. Thus, it suffices to show that

$$\Pi_{\mathcal{G} \cap \mathcal{F}_t} \lor \Pi_{\mathcal{G}' \cap \mathcal{F}_t} \subset \Pi_{\mathcal{G} \lor \mathcal{G}'} \land \Pi_{\mathcal{F}_t}$$

for all $t \in \mathcal{T}$.

Fix some $t \in \mathcal{T}$ and an event $A \in \Pi_{\mathcal{G} \cap \mathcal{F}_t} \lor \Pi_{\mathcal{G}' \cap \mathcal{F}_t}$. By definition of the supremum, $A = B \cap B'$ for some sets $B \in \Pi_{\mathcal{G} \cap \mathcal{F}_t}$ and $B' \in \Pi_{\mathcal{G}' \cap \mathcal{F}_t}$. Since $\mathcal{G}$ is connected, $\Pi_{\mathcal{G} \cap \mathcal{F}_t} \setminus \Pi_{\mathcal{F}_t} \subset \Pi_{\mathcal{G} \setminus \mathcal{F}_t} = \Pi_{\mathcal{G}}$. Equivalently,

$$\Pi_{\mathcal{G} \cap \mathcal{F}_t} \subset \Pi_{\mathcal{G} \cup \mathcal{F}_t}. \quad (5.12)$$

An analogous argument holds for $\mathcal{G}'$. By (5.12), there are two cases to consider:
If $B \in \Pi_G$ and $B' \in \Pi_{G'}$, then
\[ B \cap B' \in (\Pi_G \lor \Pi_{G'}) \cap \mathcal{F}_t = \Pi_{G \lor G'} \cap \mathcal{F}_t \subset \Pi_{G \lor G'} \land \Pi_{\mathcal{F}_t}. \]

If $B \in \Pi_{\mathcal{F}_t}$ (or $B' \in \Pi_{\mathcal{F}_t}$), then $B \cap B' \in \Pi_{G \cap \mathcal{F}_t} \lor \Pi_{G' \cap \mathcal{F}_t} \leq \Pi_{\mathcal{F}_t}$ implies that $B = B \cap B'$. But then
\[ B \cap B' = B \in \Pi_{\mathcal{F}_t} \cap \mathcal{G} \subset \Pi_{\mathcal{F}_t} \cap (\mathcal{G} \lor \mathcal{G'}) \subset \Pi_{\mathcal{F}_t} \land \Pi_{G \lor G'}. \]

By Lemma 23, it is enough to prove that the class of connected algebras is a lattice. Take the supremum $\mathcal{G} \lor \mathcal{G'}$ of two sequentially connected algebras $\mathcal{G}$ and $\mathcal{G'}$ and an event $A \in \Pi_{(G \lor G') \cap \mathcal{F}_t} \setminus \Pi_{\mathcal{F}_t}$. By Lemma 23,
\[
\Pi_{(G \lor G') \cap \mathcal{F}_t} = \Pi_{G \lor G'} \land \Pi_{\mathcal{F}_t} = \Pi_{G \cap \mathcal{F}_t} \lor \Pi_{G' \cap \mathcal{F}_t}.
\]

Thus there exist sets $B \in \Pi_{G \cap \mathcal{F}_t}$ and $B' \in \Pi_{G' \cap \mathcal{F}_t}$ such that $A = B \cap B'$. If $B \in \Pi_{\mathcal{F}_t}$, then $B' \in \mathcal{F}_t$ implies $A = B \cap B' = B \in \Pi_{\mathcal{F}_t}$ contradicting the choice of $A \notin \Pi_{\mathcal{F}_t}$. A symmetric argument implies $B' \in \Pi_{G \cap \mathcal{F}_t} \setminus \Pi_{\mathcal{F}_t}$. Since $\mathcal{G}$ and $\mathcal{G'}$ are sequentially connected, $B \in \Pi_{G \cap \mathcal{F}_t}$ and $B' \in \Pi_{G' \cap \mathcal{F}_t}$ for all $\tau \geq t$. Conclude that $A = B \cap B' \in \Pi_{G \cap \mathcal{F}_t} \lor \Pi_{G' \cap \mathcal{F}_t} = \Pi_{G \lor G'} \land \Pi_{\mathcal{F}_t} = \Pi_{(G \lor G') \cap \mathcal{F}_t}$ for all $\tau \geq t$.

To show that $\mathcal{G} \land \mathcal{G'}$ is sequentially connected, take $A \in \Pi_{G \land G' \cap \mathcal{F}_t} \setminus \Pi_{\mathcal{F}_t}$. Notice that
\[
\Pi_{G \land G' \cap \mathcal{F}_t} = \Pi_{(G \cap \mathcal{F}_t) \land (G' \cap \mathcal{F}_t)} = \Pi_{G \cap \mathcal{F}_t} \land \Pi_{G' \cap \mathcal{F}_t}.
\]

But $A \in \Pi_{G \cap \mathcal{F}_t} \land \Pi_{G' \cap \mathcal{F}_t}$ if and only if for all $B_0 \in \Pi_{G \cap \mathcal{F}_t} \cup \Pi_{G' \cap \mathcal{F}_t}$ such that $B_0 \subset A$:
\[
A = \cup_{\{B_0, B_1, \ldots, B_k\}} \cup_{B \in \{B_0, B_1, \ldots, B_k\}} B,
\]

where the union is taken over all sequences $\{B_0, B_1, \ldots, B_k\}$ of subsets of $A$ such that consecutive elements intersect and belong alternatively to $\Pi_{G \cap \mathcal{F}_t}$ and $\Pi_{G' \cap \mathcal{F}_t}$.

Since $A \in (\Pi_{G \cap \mathcal{F}_t} \land \Pi_{G' \cap \mathcal{F}_t}) \setminus \Pi_{\mathcal{F}_t}$, there exists a set $B_0 \in (\Pi_{G \cap \mathcal{F}_t} \cup \Pi_{G' \cap \mathcal{F}_t}) \setminus \Pi_{\mathcal{F}_t}$ such that $B_0 \subset A$. Fix such a set $B_0$ and consider a sequence...
\{B_0, B_1, \ldots, B_k\}$ satisfying the conditions above. For $i < k$, each $B_i$ intersects two disjoint subsets of $\mathcal{F}_i$ and so $B_i \notin \Pi_{\mathcal{F}_i}$. Moreover, if $B_k \in \Pi_{\mathcal{F}_k}$ then $B_k \cap B_{k-1} \neq \emptyset$ implies that $B_k \subset B_{k-1}$. Conclude that $A$ can be written as the union over sequences $\{B_0, B_1, \ldots, B_k\}$ in $(\Pi_{g \cap \mathcal{F}_i} \cup \Pi_{g' \cap \mathcal{F}_i}) \setminus \Pi_{\mathcal{F}_i}$. Since $\mathcal{G}$ and $\mathcal{G}'$ are sequentially connected, $A$ can be written as the union over sequences $\{B_0, B_1, \ldots, B_k\}$ in $\Pi_{g \cap \mathcal{F}_i} \cup \Pi_{g' \cap \mathcal{F}_i}$ for all $t \geq t$. Conclude that $A$ must be a subset of some element in $\Pi_{g \cap \mathcal{F}_i} \cup \Pi_{g' \cap \mathcal{F}_i}$. But since $\Pi_{g \cap \mathcal{F}_i} \cup \Pi_{g' \cap \mathcal{F}_i}$ is finer than $\Pi_{g \cap \mathcal{F}_i} \cap \Pi_{g' \cap \mathcal{F}_i}$ and $A \in \Pi_{g \cap \mathcal{F}_i} \cap \Pi_{g' \cap \mathcal{F}_i}$, it must be that $A \in \Pi_{g \cap \mathcal{F}_i} \cap \Pi_{g' \cap \mathcal{F}_i}$ for all $t \geq t$.

**Proposition 24** A stopping time $g$ induces a sequentially connected algebra.

A stopping time is a function $g : \Omega \to T$ such that $[g = t] \in \mathcal{F}_t$ for all $t \in T$. The stopping time $g$ is induces the algebra $\mathcal{G}$:

$$\mathcal{G} := \{ A \in \mathcal{F}_T : A \cap [g = t] \in \mathcal{F}_t, \forall t \in T \}.$$ 

To see that $\mathcal{G}$ is sequentially connected, first prove that

$$\{ A \in \Pi_{\mathcal{F}_i} : A \subset [g = t] \} \subset \Pi_{\mathcal{G}}, \forall t \in T \quad (5.13)$$

Fix $t \in T$ and $A \in \Pi_{\mathcal{F}_i}$ such that $A \subset [g = t]$. Since $A \cap [g = t] = A \subset \mathcal{F}_t$ and $A \cap [g = t'] = \emptyset \in \mathcal{F}_t$ for all $t' \neq t$, the definition of $\mathcal{G}$ implies $A \in \mathcal{G}$. If $B \subset A \in \Pi_{\mathcal{F}_i}$, then $B \cap [g = t] = B \notin \mathcal{F}_t$ and thus $B \notin \mathcal{G}$. Conclude that $A \in \Pi_{\mathcal{G}}$.

Next fix $t < T$ and take $A \in \Pi_{g \cap \mathcal{F}_i} \setminus \Pi_{\mathcal{F}_i}$. If $A \cap [g \geq t] \neq \emptyset$, then there exists $A' \subset A$ such that $A' \in \Pi_{\mathcal{F}_i}$ and $A' \cap [g \geq t] \neq \emptyset$. But $[g \geq t] = [g < t]^c \in \mathcal{F}_i$ and $A' \in \Pi_{\mathcal{F}_i}$ imply that $A' \subset [g \geq t]$ and so $A' \in \mathcal{G}$. In turn, $A' \in \mathcal{G} \cap \Pi_{\mathcal{F}_i}$ implies $A' \in \Pi_{g \cap \mathcal{F}_i}$ contradicting the choice of $A$. Conclude that $A \subset [g < t]$.

Fix some $t' < t$ such that $A \cap [g = t'] \neq \emptyset$. Since $[g = t'] \in \mathcal{G} \cap \mathcal{F}_t$ and $A \in \Pi_{g \cap \mathcal{F}_i}$, it must be the case that $A \subset [g = t']$. This implies $A \subset \mathcal{F}_t$, for otherwise, $A \cap [g = t] = A \notin \mathcal{F}_t$ contradicts $A \in \mathcal{G}$. For any $A' \in \Pi_{\mathcal{F}_i}$ and $A' \subset A \subset [g = t']$, equation (5.13) implies that $A' \in \Pi_{\mathcal{G}}$ and so $A' \in \Pi_{g \cap \mathcal{F}_i}$. Since $A \in \Pi_{g \cap \mathcal{F}_i}$, it must be the case that $A = A' \in \Pi_{\mathcal{G}}$. But then $A \in \Pi_{\mathcal{G}} \cap \mathcal{F}_{t+1} \subset \Pi_{g \cap \mathcal{F}_{t+1}}$ as desired.

The next example translates the Gabaix and Laibson [7] procedure for simplifying decision trees in the setting of this paper and shows that it induces a sequentially connected filtration.
Example 7 (Satisficing) "Start from the initial node and follow all branches whose probability is greater than or equal to some threshold level $\alpha$. Continue in this way down the tree. If a branch has a probability less than $\alpha$, consider the node it leads to, but do not advance beyond that node."

Thus, let $\mu$ be a measure on $(\Omega, \mathcal{F}_T)$ and $\alpha \in [0, 1]$ be a threshold level. For each event $A$ and algebra $\mathcal{F}$, define $r_\mathcal{F}(A)$ to be the smallest $\mathcal{F}$-measurable superset of $A$. The collection of events $\{A_t\}$ is a satisficing procedure if:

$$A_0 = \{ \Pi_{\mathcal{F}_0} \}, \text{ and for all } t > 0$$
$$A_t = \{ A \in \Pi_{\mathcal{F}_t} : r_{\mathcal{F}_{t-1}}(A) \in \mathcal{G}_{t-1} \text{ and } \mu(r_{\mathcal{F}_{t-1}}(A) \mid r_{\mathcal{F}_{t-2}}(A)) \geq \alpha \}. $$

It is not difficult to see that $\{A_t\}$ generates a sequentially connected filtration. In fact, the filtration is induced by the stopping time:

$$[\tau = 0] = \emptyset, \text{ and for all } t > 0$$
$$[\tau = t] = \bigcup \{ A \in \Pi_{\mathcal{F}_t} : \mu(A \mid r_{\mathcal{F}_{t-1}}(A)) < \alpha \text{ and } \mu(r_{\mathcal{F}_{t-1}}(A) \mid r_{\mathcal{F}_{t-2}}(A)) \geq \alpha \}. $$

References


