Duverger's Law Without Strategic Voting

Mark Fey*

Department of Political Science
University of Rochester

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Abstract

Duverger's Law states that plurality rule in single member districts tends to produce a two-party system. Most formal explanations of this Law rely on strategic voting. In this paper we present a model that yields Duverger's Law without this assumption. Specifically, we show that even with nonstrategic or "sincere" voters, the strategic decisions of policy-motivated candidates lead to two candidate competition. This occurs because, with sincere voting, a candidate who withdraws from the race transfers her votes to nearby candidates with similar policy positions. For a candidate in last place in a multicandidate race, this means dropping out can increase the chance that the winning policy will be one she prefers. We find that all equilibria of the model involve exactly two candidates choosing to run and thus Duverger's Law holds. In addition, we find that, in equilibrium, competition occurs on only one of the two dimensions of the issue space. This result provides a theoretical underpinning for the empirical finding that political competition is essentially unidimensional.

*Harkness Hall, University of Rochester, Rochester, NY, 14618. Office: 585-275-5810. Fax: 585-271-1616. Email: markfey@mail.rochester.edu.

1 Introduction

Political scientists have long been interested in how, across countries, the choice of electoral system affects the number and type of political parties that must compete within it. The most famous result in this tradition is Duverger's Law: plurality rule in single member districts tends to produce a two-party system. One argument that Duverger (1954) used to justify his eponymous Law is that voters will be unwilling to waste their vote on a third party that has no chance to win and will instead try to make their vote count by voting for a less preferred party that has a better chance of winning.[1]

Known as "strategic voting" or "tactical voting," this behavior can be thought of as a rational response by voters to the relative likelihoods of success of various candidates. In this way, strategic voting can be explained by decision-theoretic models (McKelvey and Ordeshook, 1972; Black, 1978; Gutowski and Georges, 1993) or by equilibrium models (Cox, 1987; Palfrey, 1989; Myerson and Weber, 1993; Fey, 1997). However, empirical researchers have questioned the extent of strategic voting in actual elections (Alvarez and Nagler, 2000; Blais, 2002; Chhibber and Kollman, 2004; Andre Blais and Turcotte, 2005). For example, Alvarez and Nagler (2000) summarize their results as "leav[ing] open the question of whether voters generally do or do not behave as strategic models of politics predict." This fact calls into question those previous explanations of Duverger's Law based on strategic voting.

In this paper, we argue that Duverger's Law can be explained purely by the actions of strategic candidates, without relying on strategic voting. We show that even with nonstrategic or "sincere" voters, policy-motivated candidates are motivated to act in a way that produces two candidate competition. Our arguments are formalized by constructing a spatial voting model with sincere voters and policy-motivated candidates. The assumption of sincere voting sets this paper apart from most of the existing formal models of Duverger's Law, which have concentrated on strategic voting as the operative explanation (Palfrey, 1989; Myerson and Weber, 1993; Fey, 1997). Also, the candidates in the model are policy-motivated, as in Calvert (1985) and Duggan and Fey (2005), but are unable to credibly commit to enact policies other

[1] See Riker (1982), Duverger (1986), and Cox (1997) for thorough discussions of Duverger's Law.

than their ideal point, as in the citizen candidate models of Osborne and Slivinski (1996) and Besley and Coate (1997). However, unlike these latter models, candidates are *purely* policy-motivated—there is no cost of entry or benefit to holding office, assumptions that are also present in Feddersen et al. (1990) and Osborne (1993). Finally, Feddersen (1992) presents a model that is perhaps most closely related to ours, but its results are driven by the assumption of costly voting, which is not present in this paper.

The simple model that we present here has several notable features. We consider a discrete issue space which consists of the four corners of a square. There are many potential candidates, all of whom are purely policy-motivated and unable to make credible policy commitments. Because they are policy-motivated, the logic of Duverger's Law works on the candidates' decision calculus. That is, because voters vote sincerely, a candidate who chooses to run may very well take votes away from some other candidate that she would most like to see win, while not affecting the vote share of candidates that she would not like to win. Put another way, in a multicandidate race, when a candidate drops out it helps candidates with similar positions and does not help candidates with dissimilar positions. In many cases, then, dropping out of the race actually makes a policy-motivated candidate who has very little chance of winning better off, because it makes it more likely a preferred policy will be enacted. In this way, the "wasted vote" logic of strategic voting operates in a similar manner on the strategic choices of potential candidates about whether to run or not.

The general idea that strategic choices of actors other than voters can contribute to the operation of Duverger's Law is, of course, present in the literature. Indeed, as a further justification of Duverger's Law, it is argued that the effect of strategic voting is magnified by the strategic actions of party elites. As Cox (1997) puts it,

> ... elite anticipation of strategic voting should lead to prudent withdrawals and hence a reduction in the number of competitors entering the field of battle. In particular, those elites who foresee that their own candidates ... will bear the brunt of strategic desertion are likely to decide that mounting a (hopeless) campaign is not worth the cost, and seek instead to throw their support behind more viable candidates ... (Cox, 151)

Note that, in this view, Duverger's Law can be explained as the political elites' reaction to or anticipation of strategic voting by the electorate and thus strategic voting is a necessary ingredient of the process by which the Law emerges. In addition, Morelli (2004) presents a model of party formation in which strategic politicians decide whether or not to become candidates, which in turn drives the formation of governing coalitions. As in this paper, Morelli observes that the strategic choices of candidates can substitute for the strategic choices of voters. However, in Morelli's model, Duverger's Law may or may not hold, depending on the values of certain parameters. In our model, on the other hand, Duverger's Law always holds.

In addition to our support for Duverger's Law, our model touches on another, seemingly unrelated issue—the dimensionality of political competition. Is politics inherently complex, with many crosscutting issues, or can this complexity be understood by summarizing it as a single dimension of "ideology"? Based on their analysis of roll-call data, Poole and Rosenthal (1985, 1991, 1997, 2001) have long argued that a single dimension can explain the vast majority of political competition. Their arguments, however, are purely empirical and they do not offer a theoretical explanation for this empirical regularity.

In this paper, we offer just such a theoretical explanation. In addition to showing Duverger's Law holds, the second main result of the paper is that, in equilibrium, competition occurs on only one of the two dimensions of the issue space. On one issue, the two candidates in the race have different positions, but on the other issue, the two candidates have the same position. The reason that this result holds is that a Duvergerian outcome in which two candidates hold opposing views on both issues invites entry by a third candidate. Thus, such a configuration cannot occur in equilibrium. In this way, our model goes beyond Duverger's Law to establish a link between the electoral system and the dimensionality of political competition.

The paper is organized as follows. The next section presents the assumptions and notation of the model. Section 3 discusses some features of Poisson games that are useful for the results, which are presented in section 4. In this section, we give two main results; exactly two candidates choose to compete in equilibrium and the two candidates compete along a single issue. The final section presents our conclusions.

2 The Model

We begin by defining the *issue space*. We suppose that there are two issues at hand and, on each issue, there are only two possible stances; for example, "For" and "Against" or "Yes" and "No". For instance, one of the issues could be whether or not to go to war with another country and the other issue could be whether or not abortion should be legal or illegal. We denote the "For" stance by 1 and the "Against" stance by 0. Let x_1 be a stance on the first issue and x_2 be a stance on the second issue. Then a stance on both issues is a pair (x_1, x_2) and we therefore have a discrete issue space consisting of the four possible positions on the two questions, $X = \{(1,1), (1,0), (0,1), (0,0)\}$. An element of X is $x = (x_1, x_2)$. The four possible positions make up a unit square. For a position x, we refer to the two positions that differ from x in only one component as *adjacent positions* and the remaining position (different on both issues) as the *opposed position* to x. Formally, if $x = (x_1, x_2)$, then the two adjacent positions to x are the positions $(1 - x_1, x_2)$ and $(x_1, 1 - x_2)$ and the opposed position to x is the position $(1 - x_1, 1 - x_2)$

Each *voter* in the electorate has an *ideal point* in X. We assume the number of voters with each ideal point is stochastic. In particular, we suppose the number of voters in the electorate is distributed according to a Poisson distribution with mean λ and that each voter is (independently) assigned an ideal point according to a probability distribution $(\pi_1, \pi_2, \pi_3, \pi_4)$ on X. We assume $\pi_1 > \pi_2 > \pi_3 > \pi_4$ and that these probabilities are assigned to the four possible positions as shown in Figure 1.[2] Thus, we refer to position 1, located at $(0, 0)$, as the most popular position, and position 4, located $(1, 0)$, as the least popular position, and so on, because position 1 is expected to have the largest share of voters and position 4 is expected to have the smallest share of voters, and so on.

Voters vote *sincerely*, that is to say they each vote for the candidate whose issue stance is closest to their own. If there are two such candidates, we assume that a voter votes for each of the two with equal probability. The winner of the election is determined by plurality rule, again with ties broken randomly with equal probability.

[2] There are two other orderings of these probabilities that are not obtainable by symmetric rotation or reflection of the assumed ordering. Although our results also hold for these orderings, for simplicity we develop our arguments for the assumed configuration only.

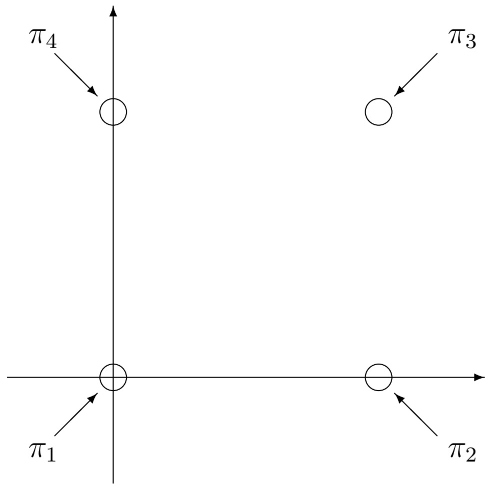

Figure 1: Distribution of a voter's ideal point

In addition to voters, we suppose that there is a set of *potential candidates*, which we denote C. The total number of potential candidates is $|C|$.[3] Each potential candidate can either enter the race (choose *In*) or stay out of the race (choose *Out*). We can think of potential candidates as having established positions before the campaign, which they cannot credibly commit to change, or that candidates cannot credibly commit to enact a policy different from their ideal point. In either case, a potential candidate's only decision is whether to enter the race or not. We denote the action *In* by 1 and the action *Out* by 0. Thus a *strategy* for potential candidate i is $s_i \in \{0, 1\}$.

Regardless of their participation decision, potential candidates care only about the policy enacted by the winner. That is, candidates are *purely policy-motivated.* We suppose that each candidate has quadratic preferences, defined in the usual way. Specifically, we suppose that a potential candidate with ideal point $\hat{x}_i \in X$ has a *utility function* given by

$$u_i(x_w) = -\|\hat{x}_i - x_w\|^2,$$

where x_w is the policy of the winning candidate. Uncertainty in the election outcome is evaluated by expected utility, as usual. We assume that, for each of the four issue

[3]Here, and in what follows, we use $|\cdot|$ to represent the number of elements in a set.

positions, there is at least one potential candidate with ideal point at that issue position. We also assume that if none of the potential candidates enters the race, then the enacted policy x_w is randomly determined according to a given probability distribution $q = (q_1, q_2, q_3, q_4)$ on X.[4]

To finish the description of the model, we assume that the potential candidates make their participation decisions simultaneously, and the election is decided by voters voting sincerely over the set of candidates who have chosen to enter the race. For a strategy profile s, we denote the set of candidates who choose *In* by $E(s)$. We define this set formally by

$$E(s) = \{i \in C \mid s_i = 1\}.$$

A (pure strategy) Nash equilibrium is a profile of strategies $s^* = (s_1^*, \ldots, s_{|C|}^*)$ such that no potential candidate has a positive incentive to change their entry decision, given the actions of the other potential candidates.[5] If a strategy profile s includes at least one potential candidate with ideal point x who enters the race, then we say that x is *occupied* under s. Otherwise, we say that x is *unoccupied* under s. Formally, the set of occupied positions given s is

$$O(s) = \{x \in X \mid \text{there exists } i \in E(s) \text{ such that } \hat{x}_i = x.\}$$

As the distribution of voters is stochastic, every candidate who participates in the election has a positive probability of winning. We denote this probability by $p_i(s)$, where we indicate the dependence of this probability on the participation decisions of the other candidates. Note that $p_i(s) > 0$ for all $i \in E(s)$ and $p_i(s) = 0$ for all $i \notin E(s)$.

3 Poisson Games

In this model, the number of voters is randomly chosen from a Poisson distribution with parameter λ. Thus, the model is a Poisson game as described by Myerson

[4]There are no restrictions placed on this distribution. In particular, we permit q to be a degenerate distribution that corresponds to a given "status quo" policy that will be enacted if no candidate runs.

[5]For simplicity, we restrict our attention to pure strategy equilibria of the model.

(1998b).[6] In this section, we review some features of these games and provide some useful approximation results.

Recall that a random variable X has a Poisson distribution with mean λ if

$$\mathrm{P}[X = k] = \frac{e^{-\lambda}\lambda^k}{x!} \quad \text{for } k = 0, 1, \ldots.$$

The Poisson distribution has two properties that we rely on in our analysis. The first property, termed *aggregation*, is that the sum of Poisson distributions is a Poisson distribution. Specifically, if $Y_1, Y_2, \ldots, Y_k$ are independent Poisson random variable with means $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively, then $Y_1 + Y_2 + \cdots + Y_k$ has a Poisson distribution with mean $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.

The second property is known as *decomposibility* or *splitting.* Suppose that the number of objects to be considered is a Poisson random variable X with mean λ and suppose each object is assigned to one of k categories according to a probability distribution $(\rho_1, \ldots, \rho_k)$, where ρ_i is the probability that the object is assigned to category i. Then, if we denote the number of objects assigned to category i by the random variable X_i, then X_i is itself distributed according to a Poisson distribution with mean $\rho_i\lambda$ and the collection of random variables $X_1, \ldots, X_k$ so defined are mutually independent. For our model, this property implies that number of voters with ideal point at position i is a Poisson random variable with mean $\pi_i\lambda$.

The results in the next section turn on the effect of the participation choice of a given potential candidate on the probabilities that candidates at various positions will win. The following proposition gives a useful approximation to these probabilities. As is standard, we use the notation $h(\lambda) \sim g(\lambda)$ for the fact that $h(\lambda)/g(\lambda) \to 1$ as $\lambda \to \infty$.

Proposition 1 *Let $(\rho_1, \ldots, \rho_M)$ be a vector such that $\rho_1 > \rho_2 > \cdots > \rho_M$, $\rho_i > 0$ for $i = 1, \ldots, M$, and $\sum_1^M \rho_i = 1$. Let $X_1, X_2, \ldots, X_M$ be independent random variables distributed according to a Poisson distribution with means $\lambda_1, \lambda_2, \ldots, \lambda_M$, where $\lambda_i = \rho_i\lambda$. Let W_i be the event that X_i is the maximum of $X_1, X_2, \ldots, X_M$. Then, for i $\neq 1$*

$$\mathrm{P}[W_i] \sim \frac{1}{\sqrt{2\pi\lambda}\, c_i} e^{-\frac{1}{2}\lambda c_i^2},$$

[6] See also Myerson (1998a, 2000) and Voorneveld (2002).

and

$$\mathrm{P}[W_1] \sim 1 - \frac{1}{\sqrt{2\pi\lambda}\; c_2} e^{-\frac{1}{2}\lambda c_2^2},$$

where

$$c_i = \frac{\rho_1 - \rho_i}{\sqrt{\rho_1 + \rho_i}}.$$

This approximation is developed in Appendix A.

Intuitively, we think of X_i as the number of votes that candidate i receives. This is a random variable, with mean λ_i. Thus, the candidate may receive an actual share of the vote that is higher or lower than λ_i. Proposition 1 shows that, in a large electorate, the probability that the *expected* winner (the most popular candidate) is the *actual* winner is very close to 1. Moreover, while the probability that the expected runner-up wins is small, it is exponentially larger than the probability that the expected third-place candidate wins, which in turn is exponentially larger than the probability that the expected fourth-place finisher wins, and so on. This is similar to the "ordering condition" of Myerson and Weber (1993) and the "magnitude theorem" of Myerson (2000).

As the results presented in the next section rely on the approximation presented in Proposition 1, our arguments are only valid for sufficiently large electorates. In particular, as λ parameterizes the expected number of voters in the electorate, we say that a strategy profile is an *equilibrium in large electorates* if there exists λ_0 such that for all $\lambda \geq \lambda_0$, the profile is a Nash equilibrium in the Poisson game with parameter λ.

4 Results

In this section we present our main results characterizing the equilibria of our model. We will show that, in equilibrium, exactly two potential candidates choose to enter the race and that they compete on only one issue. Thus, Duverger's Law holds in the model and the nature of equilibrium political competition is one-dimensional.

In order to establish our main results, we present a sequence of lemmas, whose proofs are given in Appendix B. We begin by noting that there can be no equilibrium in which only one position is occupied or in which no potential candidates choose to enter.

Lemma 1 *There is no equilibrium in large electorates such that* $|\,O(s^*)\,| \leq 1$.

It is easy to see why there can be no such equilibrium. If there were an equilibrium with just one occupied position, then a potential candidate at an unoccupied position could choose *In* and receive her ideal point with some probability, instead of the sole occupied position with probability 1. Likewise, it cannot be an equilibrium for all potential candidates to choose not to run. If no candidate runs, the outcome is randomly determined. A potential candidate would then have an incentive to enter the race and enact her ideal policy.

Next, we show that there can be no equilibrium in which some position has more than one potential candidate choosing to enter the race.

Lemma 2 *There is no equilibrium in large electorates such that* $i, j \in E(s^*)$, $i \neq j$, *and* $\hat{x}_i = \hat{x}_j$.

To understand the logic behind this lemma, consider the situation in which all four positions are occupied. In this case, a participating candidate receives votes only from voters with a matching ideal point. If a position, say x, is occupied by two or more candidates, they have the effect of splitting (equally) these votes. If one of the candidates competing at x drops out of the race, this increases the probability that a candidate at x will win, while leaving unchanged the probability that a candidate not at x will win. This same argument applies to the case of two or three occupied positions, as well. Finally, Lemma 1 rules out the case of fewer than two occupied positions. Thus, in any equilibrium, no position is occupied by more than one candidate.

Having established these simple necessary conditions, we now turn to more complex arguments. We begin by showing that there cannot be an equilibrium in which all four policy positions are occupied.

Lemma 3 *There is no equilibrium in large electorates such that* $O(s^*) = X$.

The intuition for this result is that if all four positions were occupied, the candidate at the least popular position would prefer to exit the race and help the two adjacent candidates and hurt the opposed candidate. This follows because, in a large electorate, the probability that the least popular candidate will win is very close to zero. Thus, for the least popular candidate, the "cost" of dropping out the race (the foregone chance of winning) is more than offset by the benefit gained by making it more likely that a candidate at an adjacent position will win.

Having ruled out equilibria in which all four positions are occupied, we next show that there cannot be an equilibrium in which there are three occupied positions.

Lemma 4 *There is no equilibrium in large electorates such that* $|\,O(s^*)\,| \geq 3$.

The logic operating in this lemma is very similar to the logic of Lemma 3. With three occupied positions, the least popular candidate again has an incentive to drop out of the race because her chance of winning is essentially zero, but by dropping out, she can increase the chance that the candidate at the adjacent position will win instead of the candidate at the opposed position.

Lemmas 3 and 4 capture how the logic of Duverger's Law works on the strategic calculations of the candidates in this model. Because potential candidates are purely policy-motivated, they care only about the final policy outcome, and because voters are sincere, a candidate who withdraws from the race transfers her votes to nearby candidates with similar policy positions. For a candidate in last place in a multicandidate race, this means dropping out can increase the chance that the winning policy will be one she prefers.

Combining Lemmas 1 and 4, we see that any equilibrium must have exactly two occupied positions. By Lemma 2, each of these two positions must be occupied by a single candidate. Therefore, in any equilibrium, exactly two candidates compete. We state this important result as Theorem 1.

Theorem 1 *For any equilibrium in large electorates,* $|\,E(s^*)\,| = 2$.

The substantive importance of this Theorem is that, in our model, Duverger's Law must hold. Moreover, the Duvergerian conclusion of the Theorem that only two candidates compete in the election flows not from strategic voting by the electorate but rather from the choices of strategic policy-motivated politicians about whether or not to become candidates.

It is important to note that the conclusion of Theorem 1 does not depend on the particular choice of which positions are most popular and which are the least popular. In particular, it does not depend on our assumption that the popularity of positions is ordered as in Figure 1. Rather, Theorem 1 holds for all orderings of the popularity of positions. Thus, the Duvergerian conclusion of this Theorem is a general conclusion of the model.

We next turn to a more detailed characterization of the issue positions involved in the election. We begin by showing that the two occupied positions cannot be opposed positions.

Lemma 5 *There is no equilibrium in large electorates such that* $O(s^*) = \{(x_1, x_2), (1-x_1, 1-x_2)\}$ *for some* $(x_1, x_2) \in X$*.*

The argument behind this Lemma is straightforward. Suppose two opposed positions were occupied but the other two positions were unoccupied. Then, whichever occupied position wins, the winning position will be one unit away from the unoccupied positions. This gives a potential candidate at one of the unoccupied positions an equilibrium payoff of -1. On the other hand, if this potential candidate chooses to enter the race, then she wins with some positive probability, giving utility 0, and one of the adjacent positions wins with the remaining probability, giving utility -1. Thus, no matter how small the probability that the entrant wins, this is a profitable deviation. So there can be no equilibrium with exactly two occupied and opposed positions.

Combining all of the lemmas presented so far, we conclude that the only possible equilibria of this model involve two adjacent occupied positions. Our last lemma shows that in any such equilibrium, the most popular position (corresponding to π_1) must be occupied.

Lemma 6 *There is no equilibrium in large electorates such that* $(0, 0) \notin O(s^*)$*.*

To see why the most popular position must be occupied in equilibrium, suppose there were an equilibrium with this position unoccupied. Then, by the previous lemmas,

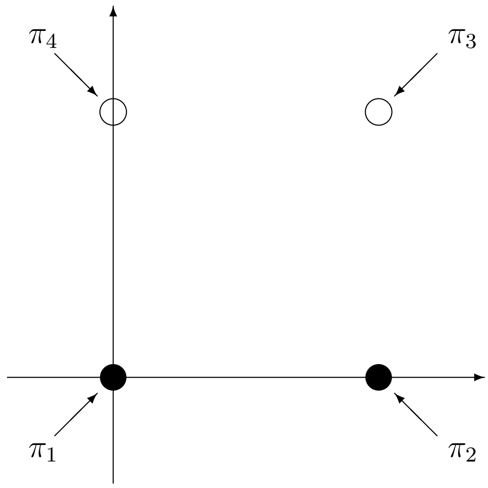

Figure 2: Unique equilibrium outcome

the two occupied positions must be adjacent. There are two possible arrangements in which this occurs. In either case, a potential candidate whose ideal point is at the most popular position could choose to enter the race and win with probability near 1. This is clearly a better outcome for this candidate and so there cannot be an equilibrium in which the most popular position is unoccupied.

We are now ready to establish that an equilibrium in large electorates exists. We also show that in any such equilibrium, the two most popular positions are occupied.

Theorem 2 *An equilibrium in large electorates exists. In any equilibrium in large electorates,* $|\,E(s^*)\,| = 2$ *and* $O(s^*) = \{(0,0),(1,0)\}$.

This result establishes that an equilibrium exists in our model. Moreover, Theorem 2 establishes that there exists a unique equilibrium outcome in our model which is illustrated in Figure 2. To be clear, although there is a unique equilibrium outcome, there are multiple equilibria which differ only by which potential candidate at each of the two most popular positions is the one that enters the race.

There are two parts to the proof of the Theorem 2. The first part establishes that a strategy profile in which two candidates enter the race, one at $(0,0)$ and one at $(1,0)$, is an equilibrium. To see this, note that neither of these two candidates wants to exit the race and no other potential candidate at one of these positions wants to enter. Moreover, by the same logic used to prove Lemma 4, no potential candidate wants to enter the race at a different position. This means that an equilibrium exists. The second part of the proof establishes that this is the unique equilibrium outcome. Given the above Lemmas, it suffices to show that there cannot be an equilibrium with one candidate at $(0, 0)$ and the other at $(0, 1)$. This is accomplished by showing that, in this case, a potential candidate at $(1, 0)$ wants to enter the race.

The substantive importance of Theorem 2 is that not only does Duverger's Law hold, but equilibrium political competition is unidimensional, even though the issue space is multidimensional. That is, the two candidates in the race choose different positions on one issue and identical positions on the other issue. Indeed, the issue that candidates choose to compete on the more "popular" of the two issues. Thus, Theorem 2 provides a theoretical rationale for models that assume candidates compete on a unidimensional issue space.

As with Theorem 1, the result in Theorem 2 that equilibrium political competition is unidimensional does not depend on our assumption that the popularity of positions is ordered as in Figure 1. This result is true no matter how voters are distributed on X. Of course, our result that candidates compete on the most "popular" issue does depend on the arrangement of ideal points. For example, if the two most popular positions were opposed, then by Lemma 5, they cannot be equilibrium positions. In this case, the equilibrium would involve the first and third most popular positions being occupied.

5 Conclusion

In this paper, we have developed a model of Duverger's Law that focuses on participation decisions by policy-motivated parties as an explanation for the Law. Thus, we have shown that elite actions are a sufficient explanation, without requiring strategic voting behavior by the electorate. This is important because some scholars have argued that voters seldom vote strategically in real elections. The results presented here suggest such a finding may not invalidate the predictions of Duverger's Law. At the same time, our model also offers an explanation as to why parties may sometimes jointly nominate a single candidate. Finally, our finding that the most popular position is always occupied in equilibrium suggests that plurality rule does well in terms of representation, even when only two candidates compete.

Much work remains to be done along these lines. In this paper, we work with a discrete two-dimensional issue space with only four possible positions. This could be generalized in several ways. One simple extension would be to consider more binary issues, so that the issue space would be the corners of a cube, for example. Another natural extension would be a model with a continuous issue space in one or more dimensions. Based on preliminary work, we suspect that versions of our results would continue to hold, but we leave this for future work.

Another aspect of the model to consider is the motivations of potential candidates. The candidates in the present model are purely policy-motivated, placing no value on holding office. In fact, the proofs of our results would permit us to relax this assumption and allow potential candidates to place a small value on holding office. At some point, however, as entry is costless, a sufficiently large value of holding office would induce potential candidates to enter and would lead to a non-Duvergerian outcome. Exactly how large the value of holding office needs to be in this case is an open question.

Finally, it would be interesting to explore a model in which some voters were strategic and some were sincere. In addition to being more realistic, such a model would offer insight into the connection between the incentives for political coordination on the mass level and on the elite level, as in Cox (1997). As it stands, the present model offers a step in this direction.

Appendix A

In this appendix, we present the approximation results supporting Proposition 1.

Let $(\rho_1, \ldots, \rho_M)$ be a vector such that $\rho_1 > \rho_2 > \cdots > \rho_M$, $\rho_i > 0$ for $i = 1, \ldots, M$, and $\sum_1^M \rho_i = 1$. Let $X_1, X_2, \ldots, X_M$ be independent random variables distributed according to a Poisson distribution with means $\lambda_1, \lambda_2, \ldots, \lambda_M$, where $\lambda_i = \rho_i \lambda$. We identify X_i as the number of voters casting votes in favor of candidate i.

We are interested in the probability that $X_i > X_j$ for all $j \neq i$.[7] Call this event W_i. We approximate this probability by the probability that $X_i > X_1$ for all $i \neq 1$ and the probability that $X_1 > X_2$ for $i = 1$. This is justified by a large deviations argument that the probability that i comes in first ahead of $k \neq 1$ is exponentially less likely than coming in ahead of 1.

We begin by noting that $\mathrm{P}[X_i > X_1] = \mathrm{P}[X_i - X_1 > 0]$ and $\mathrm{P}[X_1 > X_2] = \mathrm{P}[X_1 - X_2 > 0]$. To estimate this probability, we use the fact that the limiting distribution of a difference of Poisson distributions is normal (Johnson et al., 1992). In particular, $X_i - X_j$ is approximately distributed as $N(\lambda_i - \lambda_j, \lambda_i + \lambda_j)$. Now as we suppose $\mathrm{P}[X_i - X_1 > 0]$ is very close to zero, we can use the standard approximation for the tails of a normal distribution. This is, for large x, $1 - \Phi(x) \sim (1/x)\phi(x)$, where $\phi(x)$ and $\Phi(x)$ are the density and distribution of a unit normal, respectively. Combining all of these approximations, we find that

$$\begin{aligned} \mathrm{P}[W_i] &\sim \frac{\sqrt{\lambda_1 + \lambda_i}}{\sqrt{2\pi}(\lambda_1 - \lambda_i)} \exp\left[-\frac{(\lambda_1 - \lambda_i)^2}{2(\lambda_1 + \lambda_i)} \right] \\ &\sim \frac{1}{\sqrt{2\pi\lambda}\, c_i} e^{-\frac{1}{2}\lambda c_i^2}, \end{aligned}$$

where

$$c_i = \frac{\rho_1 - \rho_i}{\sqrt{\rho_1 + \rho_i}}.$$

The second expression in the proposition follows from the fact that $\mathrm{P}[X_1 - X_2 > 0] = 1 - \mathrm{P}[X_2 - X_1 > 0]$.

[7] As the means of the random variables are large, we ignore the possibility of a tie.

Appendix B

In this appendix, we provide proofs of the results in the text. Throughout this appendix, we label the positions in X as in Figure 1, with position 1 at $x^1 = (0, 0)$, position 2 at $x^2 = (1, 0)$, and so on. We define the set of potential candidates and entrants with ideal point at position k by C^k and $E^k(s)$, respectively.

Proof of Lemma 1: First, suppose s^* is an equilibrium with exactly one occupied position, say position x^k. Then a potential candidate $i \notin E^k(s^*)$ receives equilibrium utility $u_i(x^k) < 0$. If i deviates to entering the race, resulting in the strategy profile s', her utility is

$$p_i(s')(0) + (1 - p_i(s'))u_i(x^k).$$

As $p_i(s) > 0$ for all s, this a profitable deviation. Second, suppose s^* is an equilibrium with no occupied positions, so $E(s^*)$ is empty. In this case, x_w is determined according to the distribution $q = (q_1, q_2, q_3, q_4)$ on X. Pick k such that $q_k < 1$. In this case, a potential candidate $i \in C^k$ receives a negative payoff in equilibrium. If i deviates to entering the race, her utility is zero. This is a profitable deviation. Thus, in neither case is s^* a Nash equilibrium. ∎

Proof of Lemma 2: Suppose s^* an equilibrium in large electorates such that $i, j \in E(s^*)$, $i \neq j$, and $\hat{x}_i = \hat{x}_j$. Let $M = |\, E(s^*)\, |$ and let $m_k = |\, E^k(s^*)\, |$. We will examine in detail the case in which all four positions are occupied in equilibrium. The other cases are similar. If all positions are occupied, the total vote share split among the m_k candidates at position k is Poisson with mean $\lambda\pi_k$. Therefore, by decomposability of the Poisson, the vote share for candidate $i \in E^k(s^*)$ is Poisson with mean $\lambda\rho_i$, where $\rho_i = \pi_k/m_k$.

Now, consider a position l and a potential candidate $j \in E^l(s^*)$ such that $m_l > 1$ and $\rho_j < \max_i \rho_i$. Again, the case in which ρ_j is the maximum is similar. Then, in equilibrium, the probability that one of the candidates with ideal point at position l wins is $m_l p_j(s^*)$. Now suppose that candidate j switches her strategy to *Out*, resulting in the strategy profile s'. In this case, the voters at position l will split their votes equally between the remaining candidates at position l. Therefore, the probability that one of the candidates with ideal point at position l wins is $(m_l - 1)p_j(s')$. Let

$\rho_j' = \pi_j/(m_l - 1)$. We claim that

$$(m_l - 1)p_j(s') > m_l p_j(s^*).$$

To see this, note that by Proposition 1, for sufficiently large λ, this inequality is well approximated by

$$\begin{aligned} \frac{m_l - 1}{\sqrt{2\pi\lambda}\, c_i'}\, e^{-\frac{1}{2}\lambda(c_i')^2} &> \frac{m_l}{\sqrt{2\pi\lambda}\, c_i}\, e^{-\frac{1}{2}\lambda c_i^2} \\ e^{\frac{1}{2}\lambda[c_i^2 - (c_i')^2]} &> \frac{m_l}{m_l - 1}\frac{c_i'}{c_i}, \end{aligned}$$

where $c_i = (\rho_1 - \rho_i)/(\sqrt{\rho_1 + \rho_i})$ and $c_i' = (\rho_1 - \rho_i')/(\sqrt{\rho_1 + \rho_i'})$. As the right hand of the last inequality is a constant, if $c_i^2 - (c_i')^2 > 0$, then this inequality will hold in large electorates. But $c_i^2 > (c_i')^2$ follows directly from that fact that $\rho_j' > \rho_j$, so the claim is proven.

Therefore, if candidate j deviates to *Out*, the probability that some candidate at position l wins the election increases. It is straightforward to see that after this deviation, the probability that some other position wins does not increase. Therefore, the deviation by candidate j is profitable, so the strategy profile s^* is not an equilibrium in large electorates. ∎

Proof of Lemma 3: Suppose s^* an equilibrium in large electorates such that $O(s^*) = X$. By Lemma 2 there must be exactly one candidate at each position. Assign the candidates the number of their position and examine candidate 4. The equilibrium payoff for this candidate is

$$\begin{aligned} p_4(s^*)(0) + p_1(s^*)(-1) + {}&p_2(s^*)(-2) + p_3(s^*)(-1) \\ &= -[p_1(s^*) + p_3(s^*) + 2p_2(s^*)] \\ &= -[p_1(s^*) + (1 - p_1(s^*) - p_2(s^*) - p_4(s^*)) + 2p_2(s^*)] \\ &= -[1 + p_2(s^*) - p_4(s^*)]. \end{aligned}$$

Now suppose that candidate 4 switches her strategy to *Out*, resulting in the strategy profile s'. In this case, the voters at position 4 will split their votes between candidates 1 and 3. Thus, the three remaining candidates will have vote shares distributed Poisson with mean $\lambda_1+\lambda_4/2$ for candidate 1, λ_2 for candidate 2, and $\lambda_3+\lambda_4/2$ for candidate 3. The payoff to candidate 4 for this alternative strategy is

$$\begin{aligned} -[p_1(s') + p_3(s') + 2p_2(s')] &= -[p_1(s') + 1 - p_1(s') - p_2(s') + 2p_2(s')] \\ &= -[1 + p_2(s')]. \end{aligned}$$

Thus, this deviation by candidate 4 is profitable if

$$\begin{gathered} p_2(s^*) - p_4(s^*) > p_2(s') \\ (p_2(s^*) - p_4(s^*))/p_2(s') > 1. \end{gathered}$$

By Proposition 1, for sufficiently large λ, $p_2(s^*)/p_2(s')$ is well approximated by

$$\frac{c_2'}{c_2} e^{-\frac{1}{2}\lambda(c_2^2 - c_2'^2)}, \tag{1}$$

where $c_2 = (\pi_1 - \pi_2)/\sqrt{\pi_1 + \pi_2}$ and $c_2' = (\pi_1 + \frac{1}{2}\pi_4 - \pi_2)/\sqrt{\pi_1 + \frac{1}{2}\pi_4 + \pi_2}$. From this, we note that

$$\begin{aligned} c_2^2 - c_2'^2 &= \frac{(\pi_1 - \pi_2)^2}{\pi_1 + \pi_2} - \frac{(\pi_1 + \frac{1}{2}\pi_4 - \pi_2)^2}{\pi_1 + \frac{1}{2}\pi_4 + \pi_2} \\ &= \frac{(\frac{1}{2}\pi_4)(\pi_1 - \pi_2)^2 - (\pi_1 + \pi_2)(\frac{1}{2}\pi_4)(\pi_1 - \pi_2) - (\pi_1 + \pi_2)(\frac{1}{2}\pi_4)^2}{(\pi_1 + \pi_2)(\pi_1 + \frac{1}{2}\pi_4 + \pi_2)} \\ &= \frac{-\pi_4(\pi_1 - \pi_2)(\pi_2) - (\pi_1 + \pi_2)(\frac{1}{2}\pi_4)^2}{(\pi_1 + \pi_2)(\pi_1 + \frac{1}{2}\pi_4 + \pi_2)} < 0. \end{aligned}$$

Therefore, expression 1 goes to infinity as $\lambda \to \infty$. A similar argument shows that $p_4(s^*)/p_2(s')$ goes to zero as $\lambda \to \infty$. We conclude that a profitable deviation exists for sufficiently large values of λ, so the strategy profile s^* is not an equilibrium in large electorates. ∎

Proof of Lemma 4: Suppose s^* an equilibrium in large electorates such that $|\,O(s^*)\,| \geq 3$. By Lemma 3, $|\,O(s^*)\,| \neq 4$, so there must be exactly three occupied positions. By Lemma 2 there must be exactly one candidate at each position. Assign the candidates the number of their position. We will examine in detail the case in which the unoccupied position is position 4. The other cases are similar. Under s^*, the three occupied positions will have vote shares distributed Poisson with mean $\lambda_1 + \lambda_4/2$ for candidate 1, λ_2 for candidate 2, and $\lambda_3 + \lambda_4/2$ for candidate 3. If candidate 3 switches her strategy to *Out*, resulting in the strategy profile s', then the vote shares will be distributed Poisson with mean $\lambda_1 + \lambda_4$ for candidate 1, and $\lambda_2 + \lambda_3$ for candidate 2.

Candidate 3's equilibrium payoff is

$$\begin{aligned} p_3(s^*)(0) + p_2(s^*)(-1) + p_1(s^*)(-2) &= -[p_2(s^*) + 2(1 - p_3(s^*) - p_2(s^*))] \\ &= -[2 - p_2(s^*) - 2p_3(s^*)], \end{aligned}$$

and her payoff from deviating to s' is

$$\begin{aligned} -[p_2(s') + 2p_1(s')] &= -[p_2(s') + 2(1 - p_2(s')) \\ &= -[2 - p_2(s')]. \end{aligned}$$

Thus, the deviation is profitable if

$$\begin{gathered} p_2(s') > p_2(s^*) + 2p_3(s^*) \\ (p_2(s^*) + 2p_3(s^*))/p_2(s') < 1. \end{gathered}$$

If $\pi_2 + \pi_3 > \pi_1 + \pi_4$, then $p_2(s')$ is very close to 1 and the numerator is very close to 0, so the deviation is clearly profitable. So suppose that $\pi_2 + \pi_3 < \pi_1 + \pi_4$. Then by Proposition 1, for sufficiently large λ, $p_2(s^*)/p_2(s')$ is well approximated by

$$\frac{c_2'}{c_2} e^{-\frac{1}{2}\lambda(c_2^2 - c_2'^2)}, \tag{2}$$

where $c_2 = (\pi_1 + \frac{1}{2}\pi_4 - \pi_2)/\sqrt{\pi_1 + \frac{1}{2}\pi_4 + \pi_2}$ and $c_2' = (\pi_1 + \pi_4 - \pi_2 - \pi_3)$. Once again, we consider

$$c_2^2 - c_2'^2 = \frac{(\pi_1 + \frac{1}{2}\pi_4 - \pi_2)^2}{\pi_1 + \frac{1}{2}\pi_4 + \pi_2} - (\pi_1 + \pi_4 - \pi_2 - \pi_3)^2$$

$$
\begin{aligned}
&= \frac{(\pi_1 + \frac{1}{2}\pi_4 - \pi_2)^2 - (\pi_1 + \frac{1}{2}\pi_4 + \pi_2)(\pi_1 + \pi_4 - \pi_2 - \pi_3)^2}{\pi_1 + \frac{1}{2}\pi_4 + \pi_2} \\
&= \frac{(\pi_1 + \frac{1}{2}\pi_4 - \pi_2)^2 - (\pi_1 + \frac{1}{2}\pi_4 + \pi_2)((\pi_1 + \frac{1}{2}\pi_4 - \pi_2) + (\frac{1}{2}\pi_4 - \pi_3))^2}{\pi_1 + \frac{1}{2}\pi_4 + \pi_2} \\
&= \frac{(\pi_1 + \frac{1}{2}\pi_4 - \pi_2)^2[1 - (\pi_1 + \frac{1}{2}\pi_4 + \pi_2)]}{\pi_1 + \frac{1}{2}\pi_4 + \pi_2} \\
&\quad - \frac{(\pi_1 + \frac{1}{2}\pi_4 + \pi_2)(2(\pi_1 + \frac{1}{2}\pi_4 - \pi_2)(\frac{1}{2}\pi_4 - \pi_3) + (\frac{1}{2}\pi_4 - \pi_3)^2)}{\pi_1 + \frac{1}{2}\pi_4 + \pi_2} \\
&= \frac{(\pi_1 + \frac{1}{2}\pi_4 - \pi_2)^2[1 - (\pi_1 + \frac{1}{2}\pi_4 + \pi_2)]}{\pi_1 + \frac{1}{2}\pi_4 + \pi_2} \\
&\qquad + \frac{(\pi_3 - \frac{1}{2}\pi_4)(\pi_1 + \frac{1}{2}\pi_4 + \pi_2)(2\pi_1 + \frac{3}{2}\pi_4 - 2\pi_2 - \pi_3)}{\pi_1 + \frac{1}{2}\pi_4 + \pi_2}.
\end{aligned}
$$

Inspecting this expression and recalling that $\pi_2 + \pi_3 < \pi_1 + \pi_4$, we find that all of the terms are positive. Therefore, expression 2 goes to zero as $\lambda \to \infty$. A similar argument shows that $p_3(s^*)/p_2(s')$ goes to zero as $\lambda \to \infty$. We conclude that a profitable deviation exists for sufficiently large values of λ, so the strategy profile s^* is not an equilibrium in large electorates. ∎

Proof of Lemma 5: Suppose s^* an equilibrium in large electorates such that exactly two positions are occupied, and the occupied positions are opposed. In this case, a candidate at an unoccupied position is indifferent over the two possible winning positions and thus receives a payoff of -1 for sure. By deviating to *In*, such a candidate can win with some probability and receive utility 0, and otherwise receives utility -1. This is a beneficial deviation, and thus s^* is not an equilibrium. ∎

Proof of Lemma 6: Suppose s^* an equilibrium in large electorates such that position 1 is not occupied. By Lemmas 1, 4, and 5, it must be that either positions 2 and 3 or positions 3 and 4 are occupied. Let candidate i be a potential candidate with ideal point at position 1. The equilibrium payoff for i is clearly less than or equal to -1. Now suppose that candidate i switches her strategy to *In*, resulting in the strategy profile s'. This results in a payoff to candidate i of at least

$$p_i(s')(0) + (1 - p_i(s'))(-2) = (1 - p_i(s'))(-2).$$

Given that $\pi_1 > \pi_2 > \pi_3 > \pi_4$, it is easy to check that regardless of whether positions 2 and 3 or positions 3 and 4 are occupied, the expected vote for candidate i is strictly larger than the expected vote of either of the other two candidates in the race. Thus, by Proposition 1, for sufficiently large λ, $p_i(s')$ is approximately one. We conclude that this is a beneficial deviation and thus s^* is not an equilibrium in large electorates.

∎

Proof of Theorem 2: By Lemmas 1, 4, 5, and 6, an equilibrium in large electorates can exist only if it involves a single candidate at positions 1 and 2 or at positions 1 and 4. We will show that the former situation is an equilibrium in large electorates and the latter situation is not. So suppose s^* is such that $|\, E(s^*)\,| = 2$ and $|\, E^1(s^*)\,| = |\, E^2(s^*)\,| = 1$. Clearly, neither candidate $i \in E^1(s^*)$ nor candidate $j \in E^2(s^*)$ prefers to exit the race and no other candidate in C^1 or C^2 prefers to enter the race. Lemma 4 shows that no potential candidate in C^3 wants to enter the race and a straightforward modification of the argument in the proof of this lemma shows that no candidate in C^4 wants to enter, either. We conclude that s^* is an equilibrium in large electorates.

To establish that this equilibrium outcome is unique, we need only show that it is not an equilibrium for a single candidate to be at positions 1 and 4. Suppose not. That is, suppose s^* is an equilibrium in large elections such that $|\, E(s^*)\,| = 2$ and $|\, E^1(s^*)\,| = |\, E^4(s^*)\,| = 1$. Then under s^*, the candidate at position 1 will have a vote share distributed Poisson with mean $\lambda_1 + \lambda_2$ and the candidate at position 4 will have a vote share distributed Poisson with mean $\lambda_4 + \lambda_3$. If some candidate at position 2 switches her strategy to *In*, resulting in the strategy profile s', then the vote shares will be distributed Poisson with mean λ_1 for the candidate at position 1, $\lambda_2 + \lambda_3/2$ for the candidate at position 2, and $\lambda_4 + \lambda_3/2$ for the candidate at position 4.

The equilibrium payoff of the candidate at position 2 is

$$\begin{aligned} p_1(s^*)(-1) + p_4(s^*)(-2) &= (1 - p_4(s^*))(-1) - 2p_4(s^*) \\ &= -1 - p_4(s^*), \end{aligned}$$

and her payoff from deviating to s' is

$$p_2(s')(0) + p_1(s')(-1) + p_4(s')(-2) = (1 - p_2(s') - p_4(s'))(-1) - 2p_4(s')$$

$$= -1 - p_4(s') + p_2(s').$$

Thus, the deviation is profitable if

$$p_2(s') - p_4(s') > -p_4(s^*).$$

Indeed, it is sufficient to show that $p_2(s') > p_4(s')$. But this follows directly from the fact that the expected vote share for the candidate at position 2 under s' is larger than the expected vote share for the candidate at position 4. Therefore, s^* is not an equilibrium in large electorates and the proof is complete. ∎

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